The Determination of the Solar Parallax from Transits of Venus in the 18th Century

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Abstract

The transits of Venus in 1761 and 1769 initiated the first global observation campaigns performed with international cooperation. The goal of these campaigns was the determination of the solar parallax with high precision. Enormous efforts were made to send expeditions to the most distant and then still unknown regions of the Earth to measure the instants of contact of the transits. The determination of the exact value of the solar parallax from these observations was not only of scientific importance, but it was expected to improve the astronomical tables which were used, e.g., for navigation. Hundreds of single measurements were acquired. The astronomers, however, were faced by a new problem: How is such a small quantity like the solar parallax to be derived from observations deteriorated by measuring errors? Is it possible to determine the solar parallax with an accuracy of 0.02" as asserted by Halley? Only a few scientists accepted this challenge, but without adequate processing methods this was a hopeless undertaking. Parameter estimation methods had to be developed at first. The procedures used by Leonhard Euler and Achille-Pierre Dionis Duséjour were similar to modern methods and therefore superior to all other traditional methods. Their results were confirmed by Simon Newcomb at the end of the 19th century, thus proving the success of these campaigns.

Key words: *History of astronomy, 18th century astronomy, celestial mechanics, positional astronomy, transits of Venus, data processing methods, development of least squares adjustment, determination of the solar parallax, Leonhard Euler, Achille-Pierre Dionis Duséjour.*

Résumé

La détermination de la parallaxe solaire à partir des transits de Vénus au 18e siècle. *Les transits de Vénus de 1761 et 1769 ont initié les premières campagnes d'observations astronomiques globales effectuées dans un cadre de collaboration internationale. L'objectif de ces campagnes était la détermination précise de la parallaxe solaire. D'énormes efforts ont été investis pour envoyer des expéditions aux endroits les plus retirés et alors peu explorés du monde pour mesurer les instants de contact des transits. L'intérêt de la détermination de la valeur exacte de la parallaxe solaire n'était pas uniquement d'ordre scientifique, mais visait aussi à améliorer les tables astronomiques qui servaient, par exemple, à la navigation. Toutefois, les astronomes étaient mis face à un problème nouveau: comment déterminer une si petite valeur telle que la parallaxe solaire à partir de mesures entachées d'erreurs de mesure? Était-il possible de déterminer cette valeur avec une précision de 0.02" comme l'avait affirmé Halley? Seuls quelques savants relevèrent ce défi mais, en l'absence de méthodes adéquates de traitement de données expérimentales, ces tentatives étaient vouées à l'échec. Les méthodes d'estimation de paramètres devaient encore être développées. Les processus utilisés par Leonhard Euler et Achille-Pierre Dionis Duséjour ressemblaient aux méthodes modernes et étaient, de ce fait, supérieures à toutes les autres méthodes traditionnelles en usage à l'époque. Leurs résultats furent confirmés par Simon Newcomb à la fin du 19e siècle, démontrant ainsi le succès indéniable de ces campagnes.*

Mots-clefs: *Histoire de l'astronomie, astronomie du 18e siècle, mécanique céleste, astronomie de position, transits de Vénus, méthodes de réduction de données, développement de l'ajustage par moindres carrés, détermination de la parallaxe solaire, Leonhard Euler, Achille-Pierre Dionis Duséjour.*

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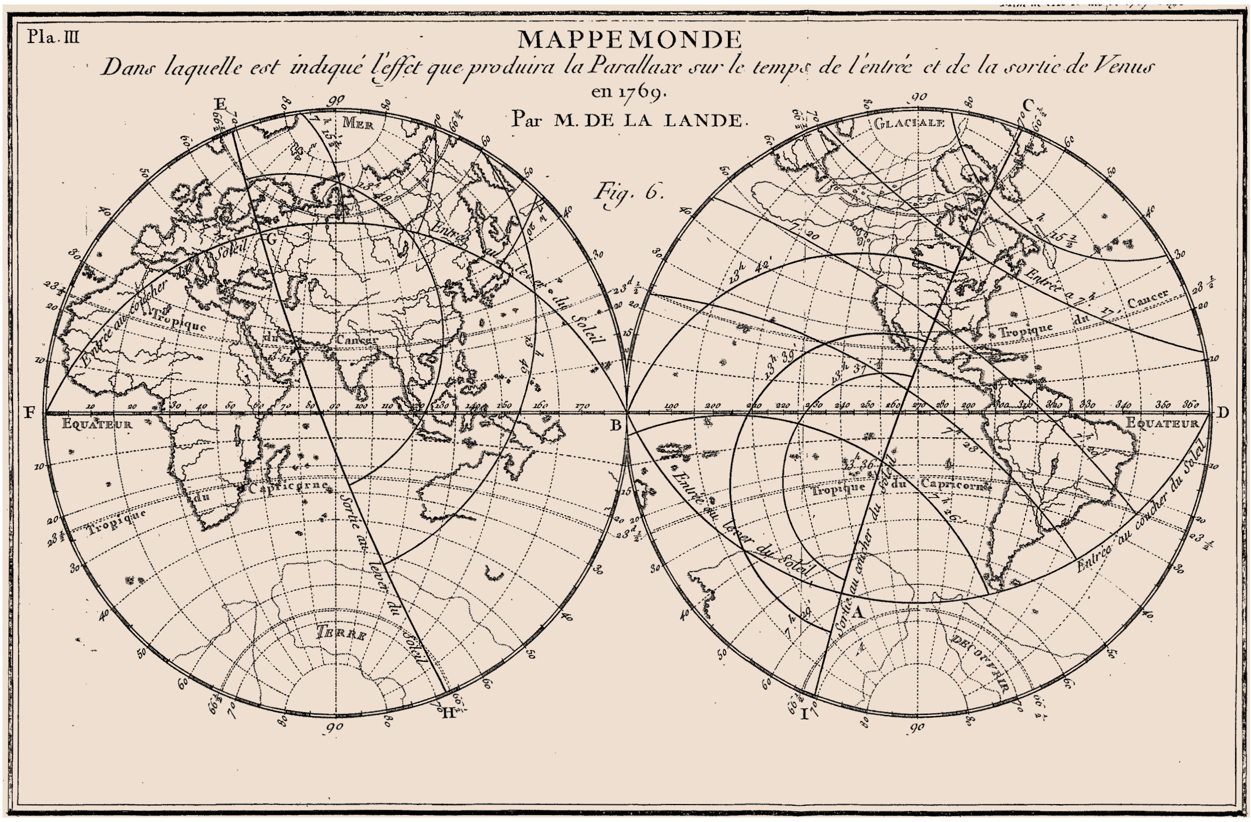


Fig. 1: A «Mappemonde» published in 1757 by Lalande, showing the «zones of visibility», i.e., the regions on Earth from where the 1769 transit of Venus (or phases of it) might be observed. (Image: A. Verdun)

Fig. 1: Mappemonde publié en 1757 par Lalande, montrant les «zones de visibilité», c-à-d les régions sur Terre d'où le transit de Vénus de 1769 (ou des phases de ce dernier) serait observable. (Image: A. Verdun)

Halley's proclamation and «Halley's method»

In 1662 Johannes Hevelius (1611-1687) published his book *Mercurius in Sole visus Anno 1661*[1]. The appendix of this book, entitled as *Venus in Sole visa*, is the first published document containing observations of a transit of Venus. It concerns the transit of December 4, 1639, which was observed by Jeremiah Horrox (1619-1641) and William Crabtree (1620-1652). Only one year later, James Gregory (1638-1675) published his book *Optica promota*[2] in which numerous astronomical problems are treated. In Problem Nº 87 he described how to determine the parallax of one of two planets being in conjunction. In a Scholium to this problem the author wrote that «*This problem has a very beautiful application, although perhaps laborious, in observations of Venus or Mercury when they obscure a small portion of the sun; for by means of such observations the parallax of the sun may be investigated.*»[3] This idea may, however, already have been formulated by Johannes Kepler (1571-1630).

Edmond Halley (1656-1742) was staying at the isle of St. Helena in 1677 and was compiling his star catalogue of the southern hemisphere when he observed the transit of Mercury on October 28, 1677 (old style). From the measured duration of this transit of $5^h\,14^m\,20^s$ he determined the (theoretical) duration of the transit with respect to the Earth's centre using the astronomical tables[4] by Thomas Streete (1622-1689). From the ratio of these values and from the values on which the tables were based Halley calculated the solar parallax obtaining 45". He might have recognized already at this time that the solar parallax could be determined even better by comparing measured durations of transits of *Venus* (instead of Mercury) observed from different places on Earth, because Venus' apparent parallax is much larger than that of Mercury. This is why he made in Volumes 17[5] and 29[6] for the years 1691 and 1716 of

1 Cf. Hevelius (1662).
2 Cf. Gregory (1663).
3 ibidem. p. 130. «*Hoc Problema pulcherrimum habet usum, sed forsan laboriosum, in observationibus Veneris, vel Mercurii particulam Solis obscurantis: ex talibus enim Solis parallaxis investigari poterit. Hactenus loquuti sumus de parallaxibus respectu globi terrestris: sequuntur quaedam de parallaxibus magni orbis terrae.*»
4 Cf. Streete (1661).
5 Cf. Halley (1694).
6 Cf. Halley (1717).

the *Philosophical Transactions* an appeal to the future generations of astronomers to use the transits of Venus of 1761 and 1769 for the determination of the solar parallax. What Halley could not know at the time of his proclamation was the fact that until the 1760ies the development of theoretical astronomy (particularly of celestial mechanics) was pushed forward in such a way that the precise determination of the solar parallax became an increasingly urgent problem to be solved and that each opportunity (e.g., transits of Mercury or conjunctions of planets, particularly of Mars) was exploited to tackle this problem. Of course, optimal success to achieve this goal was expected for transits of Venus. The solar parallax should, in particular, be determined with an accuracy of 1/500 resp. 0.02" by the use of transits of Venus, as Halley showed by a simple estimate and as «Halley's method» anticipated.

Halley describes his «method» in «the most detailed way» in his proclamation of 1717[7]. Somebody expecting a well defined method (as it is asserted by the word «methodus» in the title of his treatise) to determine the solar parallax (e.g., using this procedure similar to a «recipe»: take observations → use method → obtain solar parallax) will be disappointed. Halley certainly describes what has to be measured, namely the duration of a transit, observed from different carefully chosen places on Earth, but he did not explain or even suggest how these observations should be performed and – most of all – how these observations should be processed. «*There remains therefore Venus's transit over the sun's disk, whose parallax, being almost 4 times greater than that of the sun, will cause very sensible differences between the times in which Venus shall seem to pass over the sun's disk in different parts of our earth. From these differences, duly observed, the sun's parallax may be determined, even to a small part of a second of time; and that without any other instruments than telescopes and good common clocks, and without any other qualifications in the observer than fidelity and diligence, with a little skill in astronomy. For we need not be scrupulous in finding the latitude of the place, or in accurately determining the hours with respect to the meridian; it is sufficient, if the times be reckoned by clocks, truly corrected according to the revolutions of the heavens, from the total ingress of Venus on the sun's disk, to the beginning of her egress from it, when her opaque globe begins to touch the bright limb of the sun; which times, as I found by experience, may be observed even to a single second of time.*»[8] He might have been very much aware of the difficulty associated with the observation and processing methods. «*And by this contraction alone we might safely determine the parallax, provided the sun's diameter and Venus's latitude were very accurately given; which yet we cannot possibly bring to a calculation, in a matter of such great subtlety*.»[9] In particular, he seemed to have recognized that for the determination of the solar parallax very accurate astronomical tables would be indispensable from which certain parameters (e.g., the value for the apparent solar diameter or the ecliptical latitude of Venus) could be extracted. These parameters are then used to calculate the «contractions», i.e., the differences between the measured durations of a transit observed at the various sites and reduced to the centre of the Earth. In the final part of his treatise Halley calculates the visibility of the transit of Venus of 1761 for various places on Earth using a graphical procedure as indicated in the Figure on the copper plate attached to his paper. This procedure, however, is rather inaccurate and the results were not of great use. Halley's comments were indeed not very useful for the future astronomers, because it is out of the question that his «method» might ever have been used as a straight-forward data processing technique. Even the idea or principle of measuring the durations of a transit at well selected places on Earth can not be regarded as an «operational» observation method considering the difficulties associated with the execution of the measurements. Anyway, the «method» as stated by Halley became commonly known as «Halley's method». It will be shown, amongst others, that this «method» was far too inadequate for the determination of the solar parallax with the expected accuracy, because the problem actually was not the underlying principle, but the insufficiency of the processing methods which were used by almost all scientists in that time.

7 Cf. Halley (1717).

8 ibidem, p. 457. «*Restat itaque Veneris transitus per Solis discum, cujus parallaxis quadruplo fere major Solari, maxime sensibiles efficiet differentias, inter spatia temporis quibus Venus Solem perambulare videbitur, in diversis Terrae nostrae regionibus. Ex his autem differentiis debito modo observatis, dico determinari posse Solis parallaxin etiam intra scrupoli secundi exiguam partem. Neque alia instrumenta postulamus praeter Telescopia & Horologia vulgaria sed bona: & in Observatoribus non nisi fides & diligentia, cum modica rerum Astronomicarum peritia desiderantur. Non enim opus est ut Latitudo Loci scrupulosè inquiratur, nec ut Horae ipsae respectu meridiani accurate determinentur: sufficit, Horologiis ad Caeli revolutiones probe correctis, si numerentur tempora à totali Ingressu Veneris infra discum Solis, ad principium Egressus ed eodem; cum scilicet primum incipiat Globus Veneris opacus limbum Solis lucidum attingere; quae quidam momenta, propria experientia novi, ad ipsum secundum temporis minutum observari posse.*»

9 ibidem, p. 459. «*Atque ex hac contractione solâ liceret de parallaxi quam quaerimus tutò pronunciare, si modo darentur Solis diameter Venerisque Latitudo in minimis accuratae; quas tamen ad computum postulare, in re tam subtili, haud integrum est.*»

Transit	Observers	Stations	Nations	Expeditions	Refractors	Achromates	Reflectors
1761	> 120	> 62	9 (F)	8	66	3	40
1769	> 151	> 77	8 (GB)	10	> 50	27	49

Table 1: Manpower and equipment associated with the expeditions of the 1761 and 1769 transits of Venus (Source: Woolf 1959)

Table 1: Ressources humaines et équipements relatifs aux expéditions de 1761 et 1769 pour les transits de Vénus (Source: Woolf 1959)

The observation campaigns

The transits of Venus of 1761 and 1769 gave rise to the first global observation campaigns with international participation. Enormous efforts were undertaken of hitherto incomparable extent to sent expeditions in distant and then partly unknown regions of the Earth with the task of measuring the instants of internal and external contacts of Venus in transit. The reason for this immense effort was the determination of the value of the solar parallax with high accuracy. This was important not only for science but it was, among others, expected to improve, e.g., navigation by this result. In those times navigation on sea was performed by measuring lunar distances, i.e., angular distances between the Moon and the stars. The observed angles were then compared with the corresponding values taken from astronomical tables. The differences between observed and tabulated values were a measure for the geographical longitudes. The astronomical tables, however, were constructed with theories of the motions of Sun and Moon which are based on the solar parallax, i.e., the distance between the Earth and the Sun (the so-called *Astronomical Unit* AU) and thus depended implicitly on this important constant. Knowing the AU (e.g., expressed in a commonly used unit of length) and using Kepler's third law allows to determine the dimensions of the solar system, i.e., all distances between the solar system bodies. Just this scaling of the solar system was of tremendous scientific importance and interest. Accordingly, the relevance of the campaigns was undisputed. Not only was political and scientific prestige associated with the success or failure of these expeditions, but the fates of so many persons who had given their lives for these missions, as well. Although historically very interesting the many descriptions of the individual expeditions (sometimes tragic and sometime amusing) written by their participants and published in uncountable popular and scientific reports as well as summarized in the excellent study by Harry Woolf[10] are not considered here. Table 1 shows the truly gigantic dimensions of these undertakings for those times, at least the matters concerning manpower and equipment. The mere manufacturing of the required instruments ordered by numerous governments effectively increased the development of optical factories, particularly in England. The demand for telescopes and clocks could hardly be met. The enormous increase of achromatic telescopes used for the transit of 1769 is striking.

While the expeditions for the transit of 1761 were dominated mainly by French scientists, the leading nation of the expeditions for the transit of 1769 was Great Britain. Promoter and organizer of the expeditions on the national as well as the international level was Joseph Nicolas Delisle (1688-1768). He was responsible for the relations necessary for international co-operations, he calculated suitable observation sites and published for the first time a so-called *Mappemonde*, i.e., a world map from which the visibility zones could easily and quickly be ascertained, and he invented a procedure that later became known as the «method of Delisle» representing an alternative to Halley's method. Delisle recognized a serious disadvantage in Halley's method. The probability to observe the whole transit from one and the same place on Earth was rather small due to the local weather conditions. If the geographical longitudes of the observation sites could be determined in addition to the instants of contact and the geographical latitudes, then single contact measurements made at different sites could also be processed according to Delisle's idea. Therefore it was decided to use this instead of Halley's method, and the expeditions consequently were instructed to determine (in addition to the instants of contact) the geographic positions of the observation sites with highest priority and accuracy. Observers and (human) computers were thus both confronted with almost insurmountable problems:

1. *Calculation and selection of best-possible observation sites*. Prerequisite for the calculation of candidate observation sites were precise astronomical tables used to determine the elements of the transit as input parameters for the calculation of the visibility zones. Because of the fact that in those times this was a demanding task from the computational point of view this problem often was solved by graphical methods.

[10] Cf. Woolf (1959). Unfortunately, this book is out of print since many years and has also become rare, especially the first printing by Princeton University Press. The author said that the only copy of the book he was able to purchase in the years since its publications was a copy discovered in a bookstore in Nigeria.

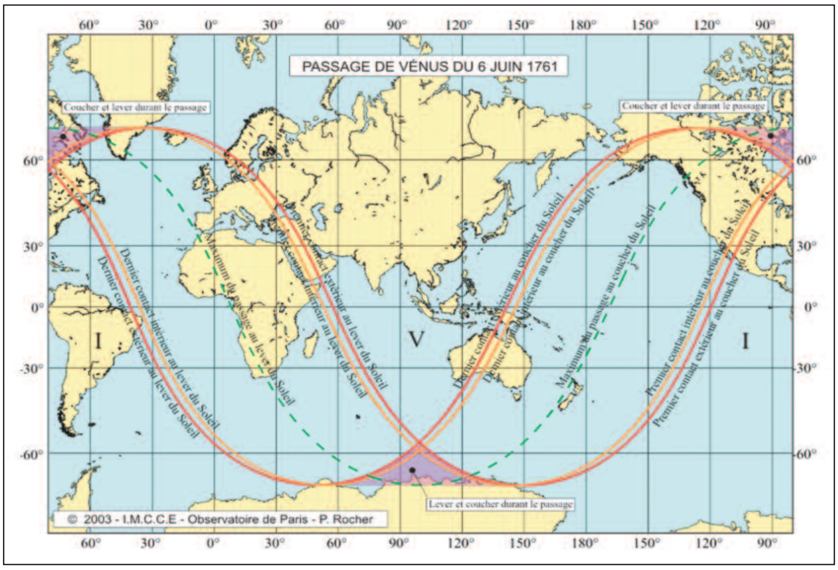


Figure 2: The visibility zones of the 1761 transit of Venus.
(Image: P Rocher, Observatoire de Paris)
Figure 2: Les zones de visibilité du transit de Vénus de 1761.
(Image: P Rocher, Observatoire de Paris)

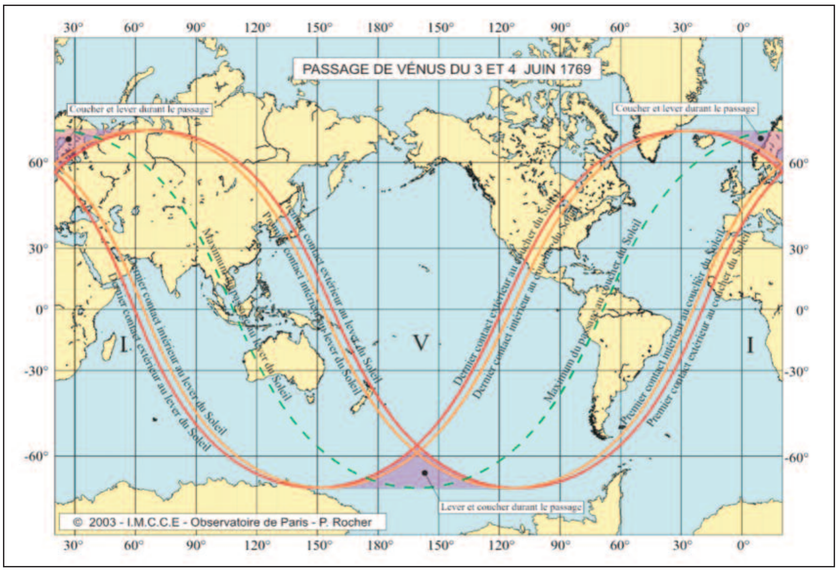


Figure 3: The visibility zones of the 1769 transit of Venus.
(Image: P Rocher, Observatoire de Paris)
Figure 3: Les zones de visibilité du transit de Vénus de 1769.
(Image: P Rocher, Observatoire de Paris)

A procedure used to produce a *Mappemonde* (Figure 1) was described, e.g., by Joseph Jérome le François de Lalande (1732-1807) in his *Astronomie*[11], one of the best textbooks then available, which was published in three editions in 1764, 1771, and 1792. This procedure was similar to the commonly used methods to determine the visibility zones for solar and lunar eclipses[12]. The observation sites had to be selected very carefully considering on the one hand that the whole transit could be observed if possible, and on the other hand that the sites were situated in a region on Earth where climate and weather conditions allowed to observe the transit successfully. With respect to these constraints it was surely not a simple task to choose the destinations in such a way that they were both situated within the visibility zones and distributed optimally over the Earth's globe. Figures 2 and 3 illustrate the global visibility zones for the transits of 1761 and 1769. Figures 4 and 5 display the places on Earth from which the transits actually were observed.

2. *Calculation of precise astronomical tables*. Such types of tables were not only used for drawing a *Mappemonde*, but especially for calculating observables (e.g., duration of the transit, instants of internal and external contacts) valid for a particular place on Earth and for the Earth's centre to which place the observations had to be reduced for comparison. The problem consists in the fact that the value of the solar parallax should be known a priori for the construction of these tables. One had therefore to presume such a value used in a model given by celestial mechanics (perturbation theory), and this model yields the orbital elements of the two planets Earth and Venus. But not only the solar parallax, but a series of so-called *astronomical constants* form – together with the model – the basis for the construction of astronomical tables. Inaccuracies of these constants have negative consequences for the precision of the elements determined by the tables. The most important astronomical tables available in those times were

11 Cf., e.g., Lalande (1792).

12 It may be an interesting and extremely instructive task for lessons in intermediate schools, in particular lessons of Geometry, projective Geometry, Mathematics or Astronomy, to study and reconstruct this geometric procedure in detail.

the planetary tables[13] by Halley (updated and edited by Chappe d'Auteroche and Lalande, Figure 6) as well as the solar tables[14] by Nicolas Louis de Lacaille (1713-1762) (Figure 7). The tables have to be updated periodically due to their inaccuracies. The transits of Mercury taking place some time before the transits of Venus offered an ideal opportunity for tuning and improving the parameters of the tables, which turned out to be crucial for the observation predictions of the forthcoming transits of Venus. Moreover, the tables had a further and likewise important function. They yield approximate values for the parameters to be estimated by the various processing methods.

3. *Performing expeditions*. The problems associated with the difficulties of the expeditions were impressively described by Woolf[15] and therefore are not discussed here in depth. It should be mentioned, however, that in the 1760ies France and England were at war – a situation increasing even more the difficulties involved with the expeditions. In fact, one of the most tragic figures of the two transits was the French scientist with the melodious name Guillaume-Joseph-Hyacinte-Jean-Baptiste Le Gentil de la Galaisière (1725-1792). He was extremely beset by inconveniencies caused by war and weather conditions. Le Gentil was sent on his journey to the French colony at Pondichéry on March 16, 1760. Shortly before approaching the harbour of Isle de France his vessel was damaged by a hurricane. He had to change ships with his entire equipment, got into very bad weather again, and was told near the coast of Malabar that meanwhile Pondichéry was captured by the British. He had to go back to Isle de France and was compelled to observe the transit of June 6, 1761, from the rocking ship at sea. Consequently those observations had no scientific use, although his measurements were performed at best weather conditions. Therefore he decided to stay in the region and to wait for the next transit of 1769 which he wanted to observe at Manila where he expected the weather conditions to be most advantageous. Having waited in Manila for a long time he received advice from the Paris academy to observe the transit at Pondichéry with special permission by the British. He respected Lalande's authority, followed his order and prepared for observation at Pondichéry. On the day of transit the weather was superb, but shortly before the beginning of the transit the sky was

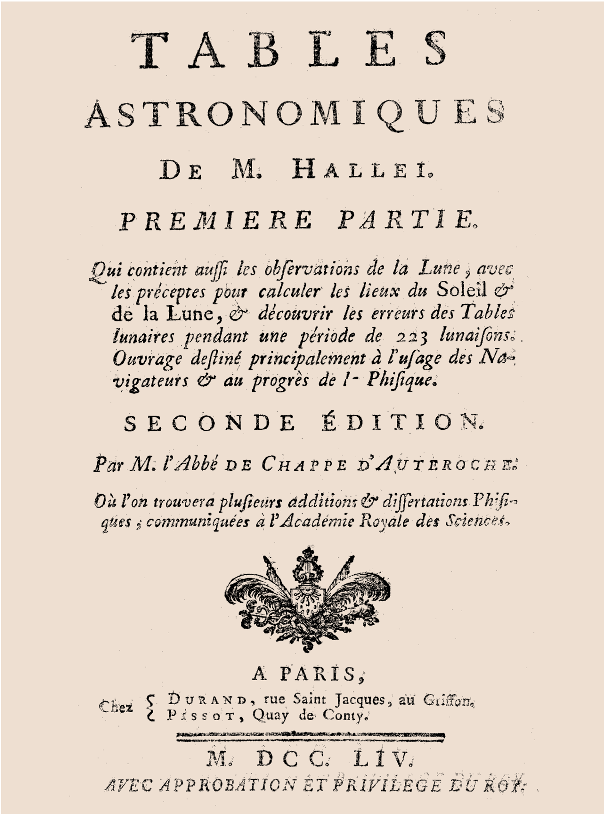
Figure 4: Sites of the installed observation stations for the 1761 transit of Venus. (Image: F Mignard, Observatoire de la Côte d'Azur)

Figure 4: Sites des stations d'observation installées pour le transit de Vénus de 1761. (Image: F Mignard, Observatoire de la Côte d'Azur)

Figure 5: Sites of the installed observation stations for the 1769 transit of Venus. (Image: F Mignard, Observatoire de la Côte d'Azur)

Figure 5: Sites des stations d'observation installées pour le transit de Vénus de 1769. (Image: F Mignard, Observatoire de la Côte d'Azur)

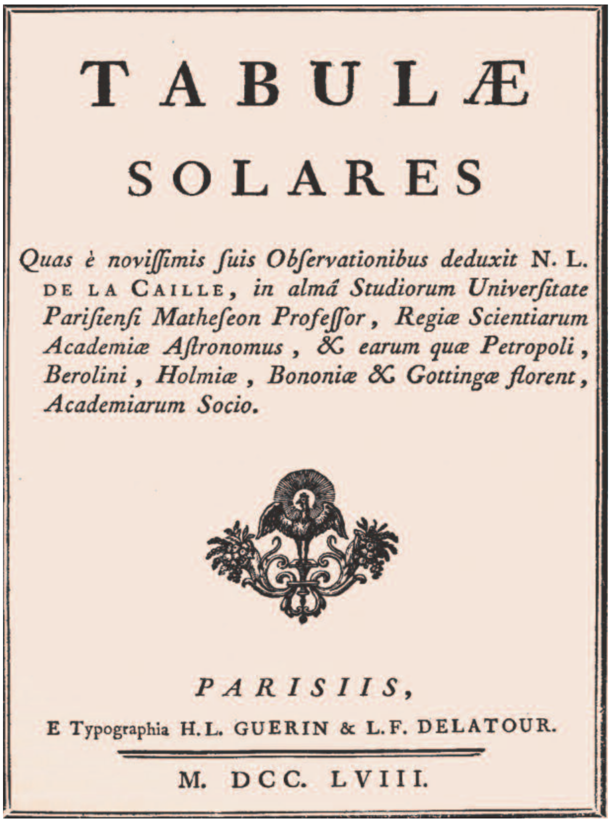
13 Cf. Halley (1754).
14 Cf. Lacaille (1758).
15 Cf. Woolf (1959).



TABLES
ASTRONOMIQUES
DE M. HALLEI.
PREMIERE PARTIE.
Qui contient aussi les observations de la Lune, avec les préceptes pour calculer les lieux du Soleil & de la Lune, & découvrir les erreurs des Tables lunaires pendant une période de 223 lunaisons. Ouvrage destiné principalement à l'usage des Navigateurs & au progrès de l'Phisique.
SECONDE ÉDITION.
Par M. l'Abbé DE CHAPPE D'AUTEROCHE.
Où l'on trouvera plusieurs additions & dissertations Phisiques, communiquées à l'Académie Royale des Sciences.
A PARIS,
Chez { DURAND, rue Saint Jacques, au Griffon. PISSOT, Quay de Conty.
M. DCC. LIV.
AVEC APPROBATION ET PRIVILEGE DU ROI.

Figure 6: Title-page of Halley's astronomical tables, updated by Chappe d'Auteroche and Lalande. (Image: A Verdun)

Figure 6: Page de titre des tables astronomiques de Halley, mises à jour par Chappe d'Auteroche et Lalande. (Image: A Verdun)



TABULÆ
SOLARES
Quas è novissimis suis Observationibus deduxit N. L. DE LA CAILLE, in almâ Studiorum Universitate Parisiensi Matheseon Professor, Regiæ Scientiarum Academiæ Astronomus, & earum quæ Petropoli, Berolini, Holmiæ, Bononiæ & Gottingæ florent, Academiarum Socio.
PARISIIS,
E Typographia H.L. GUERIN & L.F. DELATOUR.
M. DCC. LVIII.

Figure 7: Title-page of the solar tables published by Lacaille. (Image: A Verdun)

Figure 7: Page de titre des tables solaires publiées par Lacaille. (Image: A Verdun)

clouded and cleared up only after the transit had finished. During the journey back to France he learned that the weather in Manila would have been excellent. When he returned to Paris after 11 years, 6 months and 13 days he was faced by the fact that meanwhile all his possessions had been distributed among his heirs, assuming that he did not survived the expedition. The fate of Jean Chappe d'Auteroche (1722-1769) was even more severe during his expedition to San José (California) in 1769. Most of the participants of the expedition team, including Chappe, became affected by an epidemic disease and lost their lives, except for a few persons who brought the precious observations back to Europe. It is worthwhile reading the details of these expeditions to understand just how important the determination of the solar parallax must have been in those times, so that human beings were ready to suffer enormous tribulations while putting their lives at the service of science.

4. *Determining geographical longitudes of the observation stations and performing calibration measurements.* It was clear even before the beginning of an expedition that it's success would depend essentially on the precise determination of the geographical longitude of the station prepared for the observation. The positions of the stations were determined at almost every site, even at sites from where the whole transit might have been observed. The determination of geographical latitude was no problem, because it might be derived directly, e.g., from elevation measurements of the culminating Sun or of culminating stars in the local meridian (polar distances). Compared to this task the determination of the longitude was a much more difficult problem. There were three methods in use: observation of (a) eclipses of Jupiter's moons, (b) occultations of stars by the Moon, and (c) lunar distances (ecliptical or equatorial angular distances) with respect to certain stars. The difference between the measured instant of time of such an event and the corresponding instant calculated from astronomical tables, reduced to the meridian of Paris or Greenwich, yields the station's longitude. The positioning accuracy resulting from these procedures depended on the quality of the tables, i.e., on the lunar theory used to construct the tables, on the one hand and on the calibration of the clocks taken with the expeditions on the other hand. These clocks, mostly

Date of transit	Contact I	Contact II	Conjunction	Contact III	Contact IV	Separation
June 6, 1761	02:02	02:20	05:19	08:18	08:37	570.4
June 3, 1769	19:15	19:34	22:25	01:16	01:35	609.3

Table 2: Elements of the 1761 and 1769 transits of Venus. The instants of contact and the instants of conjunctions are given in Universal Time (UT), the smallest angular distances (separation) between the centres of Venus' and the Sun's disks are given in arc seconds. (Source: Espenak: Transits of Venus - Six Millennium Catalog 2000 BCE to 4000 CE, http://sunearth.gsfc.nasa.gov/eclipse/transit/catalog/VenusCatalog.html)

Table 2: Éléments des transits de Vénus de 1761 et 1769. Les instants de contact et de conjonction sont donnés en Temps Universel (UT), les plus petites distances angulaires (Separation) séparant les centres des disques de Vénus et du Soleil sont en secondes d'arc. (Source: Transits of Venus – Six Millenium Catalog 2000 BCE to 4000 CE, http://sunearth.gsfc.nasa.gov/eclipse/transit/catalog/VenusCatalog.html)

pendulum clocks or marine chronometers, had to be adjusted to the local meridian by astronomical observations for determining its daily drifts and drift rates. This was certainly a non-trivial task considering the adverse circumstances the measurements had to be performed (by day and night) and considering the fact that the fragile clocks had to be transported by sea or surface over thousands of kilometres – a torture for the materials and mechanics of the clocks.

5. *Observation of the instants of contact*. The measuring of the exact instants of contact required at least two persons: one at the telescope observing and commenting the phases of the transit, and one at the clock(s) reading and noting the instants of time of the various events in progress. In most cases, however, the observations were noted by a third person. During the preparatory phase to the expeditions there were published «recommendations» stating and defining what had to be observed and how the measurements had to be carried out. These recommendations thus probably represent the earliest documents attempting to standardize observation methods. Although it was in most cases not possible to follow these rules, the development of this idea became one of the most important pre-conditions for a central processing of astronomical data acquired at various sites. Just this aspect proved to be a crucial point particularly when processing the transit observations using «traditional» methods, because there were obviously different interpretations in measuring the instants of contact which were affected by the phenomena of the so-called black-drop-effect and influenced by individual perception. Table 2 shows for the transits of Venus in 1761 and 1769 the instants of internal and external contacts, the moments of conjunction, and the smallest angular distances between the centres of Venus' and the Sun's disk.

6. *Development of appropriate processing methods and reduction of observations*. As already mentioned above, it turned out in retrospect that it was not the quality of the observations and of the time measurements that were actually the crucial points for the determination of the solar parallax, but rather the methods used to reduce and process the data. The output of the observation stations consisted at least in hundreds of single measurements, representing – for those times – a huge amount of data out of which a very small value, the solar parallax, had to be derived. The astronomers were thus confronted with a new and almost unsolvable problem: How can parameters correctly be estimated from redundant data? In particular, how is the solar parallax to be determined with an accuracy of 0.02" according to Halley's estimation? Only a few scientists accepted this challenge, but without appropriate processing methods this was an almost hopeless attempt – probably nobody was aware of this fact, however, since the necessary parameter estimation methods were still to be developed in the future.

What was actually measured, or, which *observables* were measured? Two types of observables may be defined: primary and secondary (or derived) observables. The instants of time t_1, t_2, t_3, t_4 of the four contacts (so-called *epochs of external and internal contacts*) directly read from the (calibrated) clocks and corrected due to the drifts and drift rates of the clocks are primary observables. In most cases these instants of time were measured in true local time determined from observations of corresponding elevations of the Sun or of stars. From these instants of contact, the durations

$$\Delta t_{32} = t_3 - t_2,\ \Delta t_{41} = t_4 - t_1,\ \Delta t_{42} = t_4 - t_2$$

were derived as secondary observables. In addition, the distances between the limbs of Venus' and the Sun's disks were continuously measured (as primary observables) by a few stations during the transit using filar micrometers. The minimum distance Δz_{VS} between the centres of Venus' and the Sun's disks was derived (as secondary observable) from these measurements, where the apparent diameters of Sun and Venus were either measured as well or were taken from astronomical tables.

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TABLE des Parallaxes horizontales du Soleil, réſultantes des Obſervations du ſecond contact intérieur des bords du Soleil & de Vénus.

VILLES & AUTRES LIEUX.	LATITUDE de CES LIEUX.	DIFFÉRENCE des temps calculée.	DIFFÉRENCE des temps obſervée.	Parallaxe horizontale du Soleil, réſultante de la comparaiſon faite avec le cap B. E.	Rodrigue.	Liſbonne.
	D. M. S.	M. S.	M. S.	S.	S.	S.
Le cap de Bonne-eſpér.	33. 55. 15 M	3. 43,21	2. 11,00			
Rodrigue	19. 40. 40 M	....	....	5,87		
Liſbonne	38. 43. 23 S	2. 58,69	3. 07,0	7,91	10,46	
Tobolsk	58. 12. 22	7. 53,34	7. 51,52	8,65	9,96	9,65
Torneå	65. 50. 50	7. 08,49	7. 04,5	8,52	9,91	9,51
Cajanebourg	64. 13. 30	7. 01,92	7. 04,5	8,61	10,06	9,76
Upſal	59. 51. 50	6. 16,83	6. 33,0	8,60	10,22	9,99
Stockolm	59. 20. 30	6. 13,37	6. 06,0	8,33	9,80	9,21
Goettingen	51. 31. 54	5. 01,65	5. 08,0	8,36	10.23	9,89
Greenwich	51. 28. 30	4. 54,58	5. 13,0	8,57	10,62	10,87
Sherburn	50. 55. 00	4. 49,91	5. 02,0	8,45	10,45	10,42
Tirnau	48. 23. 30	4. 47,25	4. 55,0	8,42	10,32	10,05
Wezlas	48. 36. 30	4. 45,37	4. 35,0	8,08	9,64	8,25
Schwezingen	49. 21. 00	4. 45,43	4. 23,0	7,79	9,31	7,32
Ingolſtadt	48. 46. 00	4. 40,74	4. 33,5	8,03	9,74	8,48
Dillingen	48. 30. 06	4. 38,73	4. 41,0	8,22	10,13	9,54
Munich	48. 09. 55	4. 35,93	4. 27,0	7,97	9,68	8,22
Paris. *L.*	48. 50. 14	4. 33,56	4. 57,0	8,61	10,85	11,59
Paris. *M*	48. 50. 14	....	4. 41,0	8,29	10,27	9,91
Laubac	46. 02. 00	4. 22,23	4. 48,0	8,63	10,97	12,05
Bologne	44. 29. 36	4. 05,19	4. 28,0	8,58	10,93	13,35
Florence	43. 46. 53	3. 57,52	3. 43,0	7,69	9,39	6,12

Il paroît clair qu'il s'eſt gliſſé quelque erreur dans l'obſervation de Florence, ſon réſultat eſt trop diſparat d'avec celui des autres obſervations, je n'en ferai point uſage. Il reſte dix-huit combinaiſons, ſans compter celles des trois termes de comparaiſons. Les dix-huit combinaiſons de l'obſervation du cap, donnent toutes la parallaxe du Soleil moindre que 9 ſecondes, ou même que 8",7, & plus forte que 8 ſecondes, excepté

Figure 8: Page 478 from the treatise published by Pingré in 1763. (Image: A. Verdun)

Figure 8: Page 478 du traité publié par Pingré en 1763. (Image: A. Verdun)

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aucune des combinaiſons poſſibles, quoique j'euſſe pû abſolument négliger la dernière, c'eſt-à-dire, celle des obſervations de Paris & de Greenwich. Cette combinaiſon n'eſt pas raiſonnable, la différence des temps des obſervations étant ſi petite, que la plus légère erreur influe très-ſenſiblement ſur la détermination de la parallaxe; cette combinaiſon cependant pourroit être regardée comme confirmative des autres, puiſqu'il n'y a pas une ſeconde entière à retrancher du temps de l'obſervation de Greenwich, ou à ajoûter au temps de celle de Paris, pour en conclurre la parallaxe de 10",42.

LIEUX DONT LES OBSERVATIONS SONT COMPARÉES.	DIFFÉRENCE des temps calculée.	DIFFÉRENCE des temps obſervée.	Parallaxe horizontale du Soleil.
	M. S.	M. S.	Sec.
Stockolm, Torneå	0. 55,22	0. 58,5	10,59
Stockolm, Tobolsk	1. 40,07	1. 45,52	10,54
Greenwich, Tobolsk	2. 58,76	3. 07,52	10,49
Greenwich, Torneå	2. 13,91	2. 20,5	10,49
Tobolsk, Torneå	0. 44,85	0. 47,02	10,48
Paris, Torneå	2. 34,53	2. 41,5	10,45
Paris, Tobolsk	3. 10,78	[illegible]	10,44
Rodrigue, Torneå	7. 08,49	7. 26,5	10,42
Paris, Rodrigue	4. 33,56	4. 45	10,42
Rodrigue, Tobolsk	7. 53,34	8. 13,52	10,42
Greenwich, Stockolm	1. 18,69	1. 22	10,42
Greenwich, Rodrigue	4. 54,58	5. 06	10,39
Stockolm, Rodrigue	6. 13,27	6. 28	10,39
Paris, Stockolm	1. 39,71	1. 43	10,33
Paris, Greenwich	0. 21,02	0. 21	[illegible]

Je conclus donc que ſi les différences de longitude ſont exactement déterminées dans le Mémoire cité de M. l'abbé de la Caille, la parallaxe horizontale du Soleil eſt de 10",42 dans ſon apogée, de 10",60 dans ſes moyennes diſtances, & dans ſon périgée de 10",78.

Figure 9: Page 486 from the treatise published by Pingré in 1763. (Image: A. Verdun)

Figure 9: Page 486 du traité publié par Pingré en 1763. (Image: A. Verdun)

From the observables provided by the observation stations it was tried to determine the solar parallax using more or less adequate processing methods. The total number of single measurements resulting from the two transits of 1761 and 1769 are assumed to be 1000; the number of published observation reports and scientific treatises was much more than 100. The number of individuals, however, involved with the processing of the data was not more than 10.

Traditional processing methods

All but two scientists used principally the same processing method (disregarding some minor variations in the use of this method) consisting in the following steps:

1. Correction of the measured instants of contact (primary observables) due to clock drifts and drift rates yielding corrected observation epochs of the internal and external contacts t_1, t_2, t_3, t_4.
2. Derivation of secondary observables (e.g., durations Δt_{32} , Δt_{41} , Δt_{42} of the transit)
3. Reduction of the observables (instants of contact, durations of transit) to a certain meridian (e.g., of Paris or Greenwich) or to the Earth's centre.
4. Calculation of theoretical values for these observables for the observation epochs and for the corresponding meridian or for the Earth's centre using astronomical tables.
5. Calculation of the differences between the reduced observables of various observation stations yielding a series of difference values Δ_{Obs}.
6. Calculation of the differences between the theoretical observables of various observation stations yielding a series of difference values Δ_{Theory} .
7. Comparison and averaging of the difference values Δ_{Obs} and Δ_{Theory}, neglecting outliers if necessary, yielding a series of averaged values for Δ_{Obs} and Δ_{Theory} .
8. Determination of the «observed» solar parallax π_{Obs} for each doublet of Δ_{Obs} and Δ_{Theory} using the formula (model) $\pi_{Obs} = (\Delta_{Obs} / \Delta_{Theory})\, \pi_{Theory}$, where π_{Theory} represents the (theoretical) a priori value of the solar parallax used for the construction of the tables.
9. Averaging (arithmetic mean) of the resulting values for π_{Obs}, neglecting outliers if necessary, yielding an averaged value for π_{Obs}.

10. Scaling of π_{Obs} due to the fact that it is valid only for the date of the transit (i.e., when AU = 1.015) which does not coincide with the date when AU = 1.0000 corresponding to the mean distance between Sun and Earth. Therefore the mean solar parallax is given by the relation $\pi_\odot = 1.015\ \pi_{Obs}$.

This method is based essentially on the principle of averaging and on the assumption that the functional relation between observed and calculated value of the solar parallax (i.e., the «model» $\pi_{Obs} = (\Delta_{Obs} / \Delta_{Theory})\ \pi_{Theory}$) is linear. Some examples illustrate this kind of data processing, which was used, e.g., by Alexandre Guy Pingré (1711-1796), James Short (1710-1768), Thomas Hornsby (1733-1810), and Andrew Planman (1724-1803).

Analyses of the 1761 transit observations

In his first treatise[16] of 1763 Pingré used the solar tables of Lacaille, the Venus ephemeris of Halley, and an a priori value π_{Theory} = 10.0" of the solar parallax for the calculation of theoretic observables. In a first «method» he compares the transit durations Δt_{32} measured at 5 observation stations with the duration measured at Tobolsk, obtaining the arithmetic mean π_{Obs} = 9.93". In a second «method» he compares the calculated angular distances between the centres of Venus' and the Sun's disks Δz_{VS} at 5 stations with the corresponding value measured at Rodrigues, resulting in a mean value π_{Obs} = 10.14". A third «method» (Figure 8) compares the instants of the second contact t_2 measured at 18 stations with t_2 measured at the Cape, which yields π_{Obs} = 8.43" (mean of 16 values), at Rodrigues, which yields π_{Obs} = 10.02" (mean of 14 values), and at Lisbon, which yields π_{Obs} = 9.89" (mean of 11 values). Finally, he compares the instants of the second contact t_2 measured at 6 stations with one another (Figure 9), thus obtaining $\pi_\odot$ = 10.60" (mean of 15 values).

Short assumes π_{Theory} = 8.5" for the solar parallax in his first treatise[17] of 1762. After having averaged the measured instants of contact for each station, he then reduced in a first «method» these mean values to the meridian of Greenwich and compares the instants of the first internal contact t_2 of 15 stations with t_2 measured at the Cape, obtaining π_{Obs} = 8.47" (mean of 15 values) and π_{Obs} = 8.52" (mean of 11 values), respectively, and resulting in $\pi_\odot$ = 8.65". In a second «method» he compares the durations of the transit Δt_{32} measured at 15 observation stations with the duration measured at Tobolsk, obtaining the arithmetic mean π_{Obs} = 9.56" and π_{Obs} = 8.69" (mean of 11 values), as well as with the calculated duration for the Earth's centre, yielding π_{Obs} = 8.48" (mean of 16 values) and π_{Obs} = 8.55" (mean of 9 values).

In his second treatise[18] of 1764, Short increases both the number of values to be compared and the number of «methods», being confident to thus get even more precise results. Again, he started with π_{Theory} = 8.5" for the solar parallax. In the first «method» he compares the instants of the first internal contact t_2 of 18 stations with t_2 measured at Cajaneburg, which yields π_{Obs} = 8.61" (mean of 53 values), of 17 stations with Bologna, which yields π_{Obs} = 8.55" (mean of 45 values), and again of 18 stations with Tobolsk, which yields π_{Obs} = 8.57" (mean of 37 values). From these three mean values he determines the average π_{Obs} = 8.58". In the second «method» he compares the instants of the first internal contact t_2 of 63 stations with one another, which yields π_{Obs} = 8.63" (mean of 63 values), π_{Obs} = 8.50" (mean of 49 values), and π_{Obs} = 8.535" (mean of 37 values), respectively. The arithmetic mean of these values gives π_{Obs} = 8.55". Then he calculates the mean value of the results of these two methods, obtaining π_{Obs} = $\underline{8.565}$". In the third «method» he compares the instants of the first internal contact t_2 of 20 stations with t_2 measured at the Cape, which yields π_{Obs} = $\underline{8.56}$" (mean of 21 values), π_{Obs} = 8.56" (mean of 19 values), π_{Obs} = 8.57" (mean of 37 values), π_{Obs} = 8.55" (mean of 8 values), π_{Obs} = 8.56" (mean of 6 values), and with Rodrigues, yielding π_{Obs} = $\underline{8.57}$" (mean of 21 values), π_{Obs} = 8.57" (mean of 13 values). In the fourth «method» he compares the durations of the transit Δt_{32} measured at the stations Tobolsk, Madras, Cajaneburg, Tornea and Abo with the duration measured at Grand Mount and Tranquebar, obtaining π_{Obs} = 8.68" (mean of 12 values) and π_{Obs} = $\underline{8.61}$" (mean of 8 values), respectively. In the fifth «method» he compares the calculated angular distances between the centres of Venus' and the Sun's disks Δz_{VS} of 8 stations with the corresponding value measured at Rodrigues, resulting in a mean value π_{Obs} = $\underline{8.56}$" (mean of 8 values). Finally, he compares 12 values of Δz_{VS} calculated from 12 durations Δt_{32} measured at different observation stations with one another, which yields π_{Obs} = $\underline{8.53}$" (mean of 12 values), assuming π_{Theory} = 8.56". Now he calculates the average of the underlined mean values, which yields π_{Obs} = 8.566". The mean value calculated without the value resulting from the fourth method is π_{Obs} = 8.557". Thus he ends up with the final result of π_{Obs} = 8.56".

[16] Cf. Pingré (1763).
[17] Cf. Short (1762).
[18] Cf. Short (1764).

At the beginning of his treatise[19] of 1764 Hornsby compares the transit durations Δt_{32} measured at 12 observation stations with the duration measured at Tobolsk, obtaining the mean value π_{Obs} = 9.332" (mean of 12 values) and π_{Obs} = 9.579" (mean of 10 values), assuming π_{Theory} = 9.0". Then he compares Δt_{32} measured at Tobolsk and Cajaneburg with Δt_{32} measured at Madras, which yields π_{Obs} = 9.763". In a next attempt he compares the durations of the transit measured at 13 stations with the calculated duration as seen from the Earth's centre, resulting in π_{Obs} = 9.812" (mean of 12 values) and π_{Obs} = 9.724" (mean of 10 values). In a next «method» he compares 5 angular distances between the centres of Venus' and the Sun's disks Δz_{VS} calculated from the transit durations Δt_{32} measured at 5 observation stations with the theoretical values of the durations calculated for each of these stations (using the tables), which yields π_{Obs} = 9.920", assuming π_{Theory} = 10.0" for the solar parallax, $R_\odot$ = 15' 48.5" for the radius of the Sun's disk, $R_♀$ = 29" for the radius of Venus' disk, and correcting the difference $R_\odot - R_♀$ by –2". His fifth method consists in comparing the instants of second internal contact t_3 measured at 14 observation stations with t_3 measured at the Cape, first of all neglecting the value measured at Rodrigues due to the suspicion that this station's observations were biased by a systematic error in its time measurements of 1 minute. The result is π_{Obs} = 8.692". The comparison of these 14 observation stations with Rodrigues, but without the measurement at the Cape yields the mean value π_{Obs} = 10.419". In the next step he subtracts 1 minute from the measurements of all stations and compares the results with Rodrigues, obtaining π_{Obs} = 8.654". He is convinced of thus having proved the time measurements of Rodrigues being biased. Finally he compares the reduced instants of second internal contact measured at the remaining 13 observation stations with one another (Figure 10), resulting in π_{Obs} = 9.695" (mean of 32 values). The arithmetic mean of the values resulting from these six methods is his final result, π_{Obs} = 9.736".

It was Pingré who observed in Rodrigues and who therefore was obliged to express his view of the results. In his Mémoire[20] of 1768 he confirmed his previously determined value of the solar parallax using similar «methods», resulting in π_{Obs} = 10.10" as arithmetic mean of two «methods» having yielded π_{Obs} = 9.97" and π_{Obs} = 10.24".

In his treatise[21] of 1769 Planman used two «different methods» which yielded identical values for the solar parallax. He assumed π_{Theory} = 8.2". In the first «method» he compares the instants of contact t_2, t_3 and t_4 measured at 32 observation stations and reduced to the meridian of Paris with the corresponding values measured at the Cape and at Peking. Averaging of the results yields π_{Obs} = 8.49". In the second «method» he compares the instants of contact t_3 and t_4 measured at 10 observation stations and reduced to the meridian of Paris with the corresponding values measured at Paris and at Bologna. Averaging of the results yields again π_{Obs} = 8.49". An interesting point of his treatise is the attempt to explain the black drop phenomenon by the refraction of the solar rays in the atmosphere of Venus (Figure 11). This explanation, however curiously enough, may produce just the «opposite phenomenon», namely a bright instead of a black drop.

Table 3 summarizes the results of the 1761 transit. The values for π_{Obs} and $\pi_\odot$ printed in bold figures are those as given by the treatises mentioned above. The arithmetic mean of the 14 values for the mean solar parallax $\pi_\odot$ is given by $\pi_\odot$ = 9.35" ± 0.69", the weighted mean is $\pi_\odot$ = 9.40" ± 0.72". The large variation of these results is striking. How significant are these results? The arithmetic mean of the a priori values for the solar parallax π_{Theory} used for the astronomical tables or used for the calculation of the

[493]

Places compared.	Difference of calculated times.	Difference of obferv'd times.	Sun's parallax.
	′ ″	′ ″	″
Tobolfki and Greenwich - - - -	2 41, 34	2 54, 5	9, 734
Tobolfki and Paris - - - - - -	3 0, 36	3 15	9, 736
Tobolfki and Gottingen - - - -	2 34, 70	2 47, 5	9, 744
Tobolfki and Stockholm - - - -	1 30, 34	1 37, 0	9, 663
Tobolfki and Upfal - - - - -	1 27, 23	1 33, 5	9, 646
Tobolfki and Bologna - - - - -	3 25, 75	3 37, 5	9, 513
Tobolfki and Florence - - - - -	3 31, 27	3 42, 5	9, 525
Stockholm and Greenwich - - -	1 11, (	1 17, 5	9, 824
Stockholm and Paris - - - - -	1 30, 02	1 38, 0	9, 797
Stockholm and Bologna - - - -	1 55, 41	2 00, 5	9, 396
Stockholm and Florence - - - -	2 0, 93	2 05, 5	9, 340
Tornea° and Gottingen - - - -	1 54, 59	2 07	9, 974
Tornea° and Paris - - - - - -	2 20, 25	2 34, 5	9, 914
Tornea° and Greenwich - - - -	2 1, 23	2 14	9, 948
Cajaneburg and Greenwich - - -	1 54, 58	2 08	10, 054
Cajaneburg and Paris - - - - -	2 13, 6c	2 28, 5	10, 003
Cajaneburg and Gottingen - - -	1 47, 94	2 01	10, 088
Cajaneburg and Florence - - - -	2 44, 51	2 56	9, 628
Cajaneburg and Bologna - - - -	2 38, 99	2 51	9, 679
Upfal and Paris - - - - - -	1 33, 13	1 41, 5	9, 808
Upfal and Greenwich - - - - -	1 14, 11	1 21	9, 836
Hernofand and Paris - - - - -	1 52, 47	1 57, 5	9, 402
Hernofand and Greenwich - - -	1 33, 45	1 37	9, 342
Hernofand and Bologna - - - -	2 17, 86	2 20	9, 139
Hernofand and Florence - - - -	2 23, 38	2 25	9, 101
Abo and Paris - - - - - - -	1 42, 46	1 55, 5	10, 145
Abo and Greenwich - - - - -	1 23, 44	1 33	10, 031
Abo and Bologna - - - - - -	2 07, 85	2 18	9, 714
Abo and Florence - - - - - -	2 13, 37	2 23	9, 649
Tornea° and Bologna - - - - -	2 45, 64	2 57	9, 617
Tornea° and Florence - - - - -	2 51, 16	3 02	9, 569
Greenwich and Paris - - - - -	0 19, 02	0 20, 5	9, 700

The mean of the whole is 9″, 695.

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Figure 10: Page 493 from the treatise published by Hornsby in 1764. (Image: A. Verdun)

Figure 10: Page 493 du traité publié par Hornsby en 1764. (Image: A. Verdun)

[19] Cf. Hornsby (1764).

[20] Cf. Pingré (1768).

[21] Cf. Planmann (1769).

Author	Year	«Method»	π_{Obs}	$\pi_{\odot}$	Mean value		π_{Theory}
Pingré	1761	1	**9.93**	10.08			10.0
		2	**10.14**	10.29			10.0
		3	**8.43**	8.56			10.0
		3	**10.02**	10.17			10.0
		3	**9.89**	10.04			10.0
		4	10.44	**10.60**	9.96	↔	10.0
Short	1762	1	**8.52**	**8.65**			8.5
		2	**8.69**	8.82			8.5
		2	**8.55**	8.68	8.72	↔	8.5
Short	1763	1-6	**8.56**	8.69	8.69	↔	8.5
Hornsby	1763	1-6	**9.74**	9.89	9.89	↔	9.0
Pingré	1765	1-2	**10.10**	10.25	10.25	↔	10.0
Planman	1768	1	**8.49**	8.61			8.2
		2	**8.49**	8.61	8.61	↔	8.2
(Arithmetic) Mean value				9.35" ± 0.69"			
According to the number of methods weighted mean value				9.40" ± 0.72"			9.08" ± 0.67"

Table 3: Summary of the results achieved from the 1761 transit.
Table 3: Résumé des résultats pour le transit de 1761.

theoretical observables is given by π_{Theory} = 9.08" ± 0.67", which is very similar to the mean value resulting from all methods. It may be concluded that these traditional processing «methods» only changed the a priori value π_{Theory} slightly and accidentally (depending on the observations considered). The mean values resulting from the six methods (last but three column of Table 3) are indeed strongly correlated with the a priori values π_{Theory}, the correlation coefficient being 0.92. Considering the used «model» $\pi_{Obs} = (\Delta_{Obs} / \Delta_{Theory})\ \pi_{Theory}$ this is not an astonishing result. This finding illustrates clearly that the «methods» used to solve this parameter estimation problem were simply useless or at the least insufficient. Numerous attempts to calculate the angular distances between the centres of Venus' and the Sun's disk from the measurements of the instants of contact as illustrated by Figures 11, 12 and 13, were used – in retrospect – without success in solving this task. The problem actually did not consist in the choice of the right observables to be compared with one another or in the manner to select, reduce and average the measurements, but there was no understanding of the fact that every observable was inevitably affected by errors. The crucial step in constructing an appropriate processing method thus consists in the fact whether or not the errors stemming from observation and theory were considered and introduced into the model as additional parameters to be estimated. It is just this crucial step that was made by Euler and Duséjour in their own processing methods.

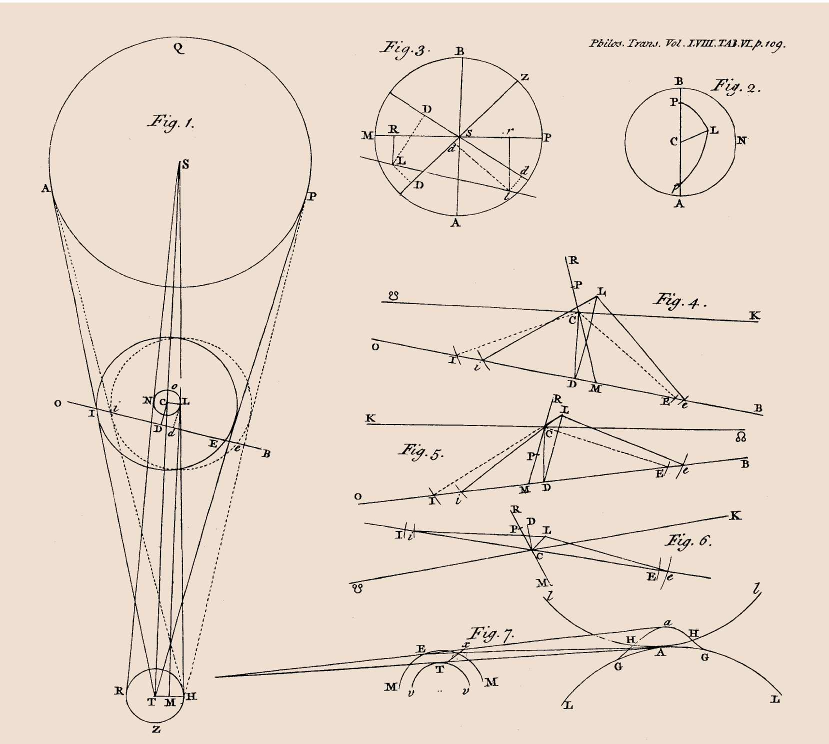


Figure 11: Copper plate figures from the treatise published by Planman in 1769. Planman tried to explain the black drop phenomenon by Figure 7 of this copper plate. (Image: A. Verdun)
Figure 11: Chalcographie du traité publié par Planman en 1769. Planman tenta d'expliquer le phénomène de la goutte noire avec la figure 7 de cette chalcographie. (Image: A. Verdun)

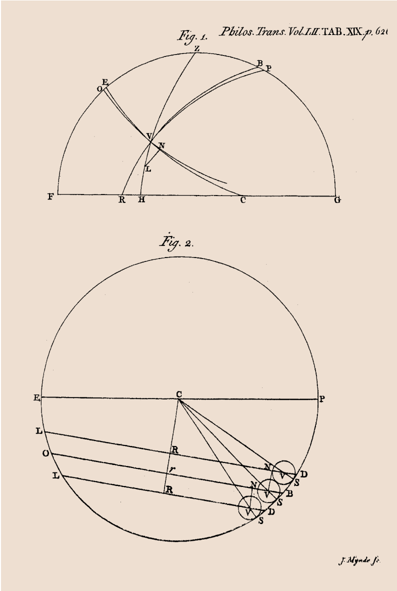


Figure 12: Copper plate of the treatise published by Short in 1762, illustrating the determination of the minimal distance between the centres of the Sun's and Venus' disks. (Image: A. Verdun)

Figure 12: Chalcographie du traité publié par Short en 1762, illustrant la détermination de la distance minimale séparant les centres des disques du Soleil et de Vénus. (Image: A. Verdun)

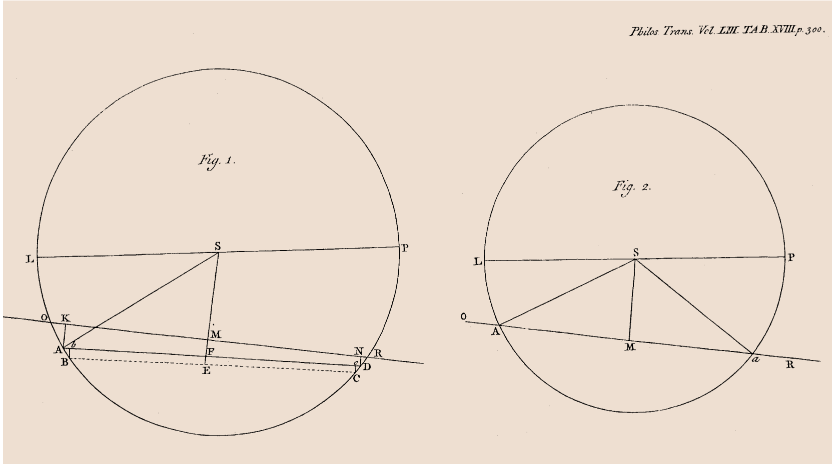


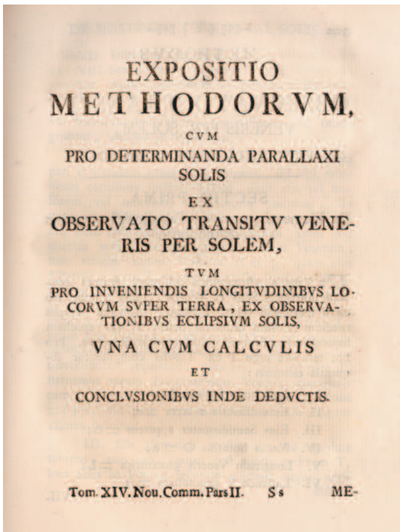
Figure 13: Copper plate of the treatise published by Short in 1764, illustrating another method of the determination of the minimal distance between the centres of the Sun's and Venus' disks. (Image: A. Verdun)

Figure 13: Chalcographie du traité publié par Short en 1764, illustrant une autre méthode pour déterminer la distance minimale séparant les centres des disques du Soleil et de Vénus. (Image: A. Verdun)

Modern parameter estimation and the methods of Euler and Duséjour

In order to judge and, consequently, to adequately recognize the value of the treatises written by Euler and Duséjour, the modern parameter estimation methods are discussed previously in their simplest form. The principle of parameter estimation consists in modelling the observations (the so-called *observables*) by mathematical formulae, in such a way that all physical laws which may be involved in the observation process are taken into account. The quantities and unknowns characterizing the model and which have to be determined are called *model parameters* or simply *parameters*. These parameters are called *estimated parameters*, because it is not possible to determine them exactly, but only with limited precision from observations which always are subject to errors introduced by the measuring process. This estimation process is called *adjustment*. Parameter estimation methods are always adjustment procedures. The goal of an adjustment consists in determining the parameters in such a way that the sum of all estimation errors equals zero. The principle of modern parameter estimation is illustrated for the case of a so-called *intermediary adjustment of linear observation equations*. It is the simplest case with respect to the so-called *adjustments with constraints* and to the *adjustments of non-linear observation equations*. These more complicated cases may, however, always be reduced formally to this simple case. One has to proceed by the following steps:

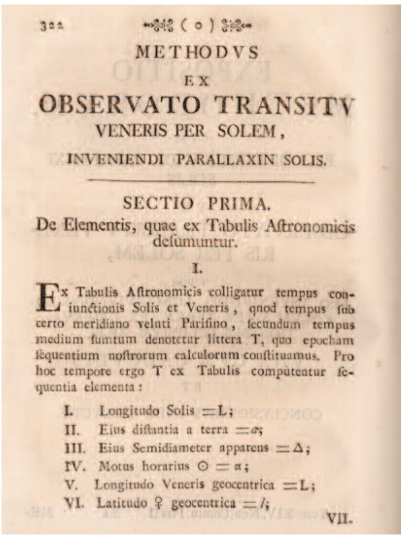
1. Formulating the so-called *observation equations*: $b = f(x_1, x_2, \ldots)$, where b represents the measured quantity, f is the functional model, and x_i are the parameters to be estimated.
2. Setting up the so-called *error equations*: $\boldsymbol{v} = \mathbf{A}\,\boldsymbol{x} - \boldsymbol{b}'$, where $\mathbf{A}$ is the model matrix representing f, $\boldsymbol{x}$ is the vector of the parameters to be estimated, $\boldsymbol{b}'$ is the observation vector representing the performed observations, and $\boldsymbol{v}$ is the residual vector representing the differences between observed and computed values of the parameters.
3. Selecting the *principle of adjustment*, e.g., the method of least squares: $\boldsymbol{v}^{\mathrm{T}}\,\mathbf{P}\,\boldsymbol{v}$ = minimal, where $\boldsymbol{v}^{\mathrm{T}}$ is the transposed of the residual vector and $\mathbf{P}$ is the weighting matrix. If $\mathbf{P}$ is equal to the unit matrix $\mathbf{E}$, then the method of least squares implies that the sum of the residuals equals to zero: $\Sigma v_i = 0$.
4. Setting up so-called *normal equations*: $\mathbf{A}^{\mathrm{T}}\,\mathbf{P}\,\mathbf{A}\,\boldsymbol{x} - \mathbf{A}^{\mathrm{T}}\,\mathbf{P}\,\boldsymbol{b}' = \mathbf{0}$. These equations result from the principle of adjustment and the error equations.
5. Determining the so-called *solution vector*: $\boldsymbol{x} = (\mathbf{A}^{\mathrm{T}}\,\mathbf{P}\,\mathbf{A})^{-1}\,\mathbf{A}^{\mathrm{T}}\,\mathbf{P}\,\boldsymbol{b}'$. The solution of this system of equations consists mainly in the problem of the inversion of matrix $\mathbf{A}^{\mathrm{T}}\,\mathbf{P}\,\mathbf{A}$. Before the computer era several procedures were developed for this task, one of which became known as the elimination procedure by Carl Friedrich Gauss (1777-1855).



EXPOSITIO
METHODORVM,
CVM
PRO DETERMINANDA PARALLAXI SOLIS
EX
OBSERVATO TRANSITV VENERIS PER SOLEM,
TVM
PRO INVENIENDIS LONGITVDINIBVS LOCORVM SVPER TERRA, EX OBSERVATIONIBVS ECLIPSIVM SOLIS,
VNA CVM CALCVLIS
ET
CONCLVSIONIBVS INDE DEDVCTIS.

Tom. XIV. Nou. Comm. Pars II. Ss ME-

Figure 14: Title-page of Euler's treatise published in 1770. (Image: A Verdun)
Figure 14: Page de titre du traité d' Euler publié en 1770. (Image: A Verdun)



322 (o)

METHODVS
EX
OBSERVATO TRANSITV
VENERIS PER SOLEM,
INVENIENDI PARALLAXIN SOLIS.

SECTIO PRIMA.
De Elementis, quae ex Tabulis Aſtronomicis deſumuntur.

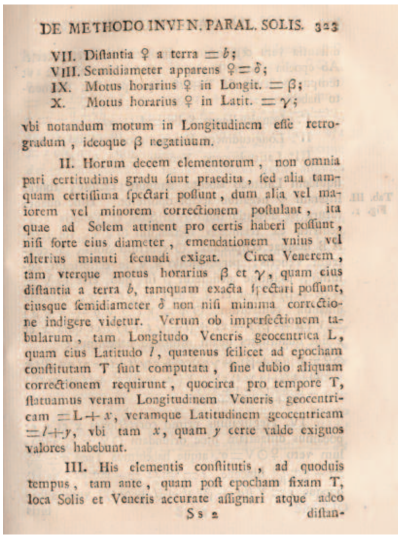
I.

Ex Tabulis Aſtronomicis colligatur tempus coniunctionis Solis et Veneris, quod tempus ſub certo meridiano veluti Pariſino, ſecundum tempus medium ſumtum denotetur littera T, quo epocham ſequentium noſtrorum calculorum conſtituamus. Pro hoc tempore ergo T ex Tabulis computentur ſequentia elementa:

I. Longitudo Solis $= L$;
II. Eius diſtantia a terra $= a$;
III. Eius Semidiameter apparens $= \Delta$;
IV. Motus horarius $\odot = \alpha$;
V. Longitudo Veneris geocentrica $= L$;
VI. Latitudo ♀ geocentrica $= l$;

VII.

Figure 15: Page 322 from Euler's treatise of 1770, illustrating the definition of the parameters. (Image: A Verdun)
Figure 15: Page 322 du traité d' Euler de 1770, illustrant la définition des paramètres. (Image: A Verdun)



DE METHODO INVEN. PARAL. SOLIS. 323

VII. Diſtantia ♀ a terra $= b$;
VIII. Semidiameter apparens ♀ $= \delta$;
IX. Motus horarius ♀ in Longit. $= \beta$;
X. Motus horarius ♀ in Latit. $= \gamma$;

vbi notandum motum in Longitudinem eſſe retrogradum, ideoque β negatiuum.

II. Horum decem elementorum, non omnia pari certitudinis gradu ſunt praedita, ſed alia tamquam certiſſima ſpectari poſſunt, dum alia vel maiorem vel minorem correctionem poſtulant, ita quae ad Solem attinent pro certis haberi poſſunt, niſi forte eius diameter, emendationem vnius vel alterius minuti ſecundi exigat. Circa Venerem, tam vterque motus horarius β et γ, quam eius diſtantia a terra b, tamquam exacta ſpectari poſſunt, eiusque ſemidiameter δ non niſi minima correctione indigere videtur. Verum ob imperfectionem tabularum, tam Longitudo Veneris geocentrica L, quam eius Latitudo l, quatenus ſcilicet ad epocham conſtitutam T ſunt computata, ſine dubio aliquam correctionem requirunt, quocirca pro tempore T, ſtatuamus veram Longitudinem Veneris geocentricam $= L + x$, veramque Latitudinem geocentricam $= l + y$, vbi tam x, quam y certe valde exiguos valores habebunt.

III. His elementis conſtitutis, ad quoduis tempus, tam ante, quam poſt epocham fixam T, loca Solis et Veneris accurate aſſignari atque adeo

Ss 2 diſtan-

Figure 16: Page 323 from Euler's treatise of 1770, continuing the definition of the parameters. (Image: A Verdun)
Figure 16: Page 323 du traité d'Euler de 1770, poursuivant la définition des paramètres. (Image: A Verdun)

Worth to mention are the stochastic errors (the so-called rms, i.e., the root mean squares) associated with the estimated parameters which may be calculated with this procedure as well and which are important indicators of the quality of both the model and the observations. Instead of the least squares adjustment usually ascribed to Gauss there is also the adjustment according to Tchebychev (adjustment is performed by minimizing the absolute value of the largest residual) as also the adjustment according to Laplace (adjustment is performed by searching for the minimal sum of the absolute values of the residuals).
To state it once and for all, neither Euler nor Duséjour nor anybody else of the 18th century used the adjustment *formally* in the way as described above. Parts of their procedures, however, closely resemble some of the steps mentioned above with respect to the goals. Particularly, the principle and objective of their methods correspond with the modern approach, namely: to estimate the parameters by minimizing the sum of the residuals, i.e., the differences between observed minus calculated quantities, so that their expectation values become close to zero, i.e., that no systematic errors remain. With respect to this goal the treatises by Euler and Duséjour are superior to all other contemporary publications concerning data processing of transit observations and thus might have been used as seminal works for future developments. This fact is illustrated by comparing their processing methods and results of the 1769 transit with those published by Hornsby and Pingré, who still used the principle of averaging.

The processing of the observation data of the 1769 transit of Venus and the determination of the solar parallax from the transits of 1761 and 1769

About one year after the transit of Venus of June 3, 1769, Euler[20] presented his results of this transit to the Academy of St. Petersburg (Figure 14). This treatise[21] contains 233 pages and was published in the same year 1770 in the second part of Volume 14 of the *Novi Commentarii*. The title of this treatise written

[20] Cf. Verdun (2003).
[21] Cf. Euler (1770).

Eclipse Date	Conjunction (UT)	Saros Type	Saros No.	Size	Latitude	Longitude	Duration
June 4, 1769	08:28	total	114	1.067	87.3 N	26.0 E	$3^m 36^s$

Table 4: Elements of the total solar eclipse of June 4, 1769. (Source: Espenak: Solar Eclipse Page, http://sunearth.gsfc.nasa.gov/eclipse/solar.html)
Table 4: Éléments de l'éclipse totale de Soleil du 4 juin 1769. (Source: Espenak: Solar Eclipse Page, http://sunearth.gsfc.nasa.gov/eclipse/solar.html)

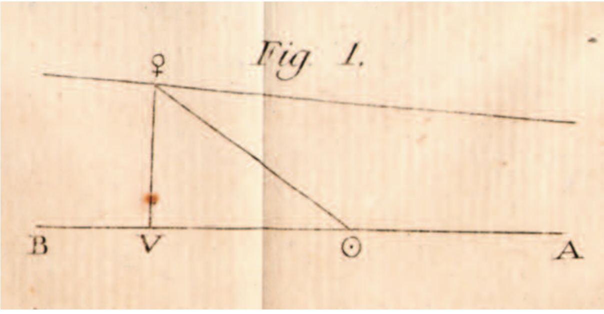


Figure 17: The first figure from Euler's treatise of 1770, defining the angular separation between the centres of the Sun's and Venus' disks. (Image: A Verdun)

Figure 17: La première figure du traité d' Euler de 1770, définissant la séparation angulaire entre les centres des disques du Soleil et de Vénus. (Image: A Verdun)

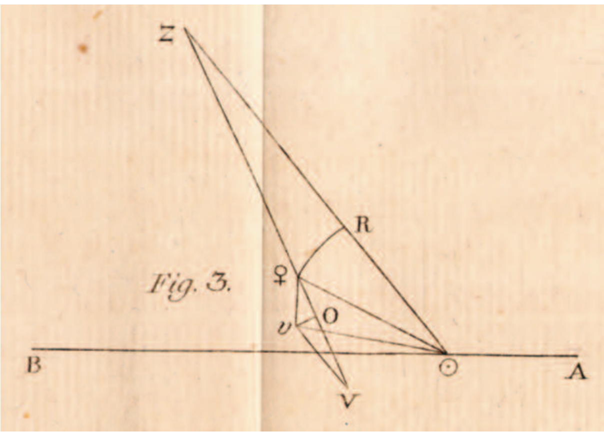


Figure 18: The third figure from Euler's treatise of 1770, defining the apparent parallactic displacement of the centre of Venus' disk due to the topographic position of the observation station with respect to the Earth's centre. (Image: A. Verdun)

Figure 18: La troisième figure du traité d'Euler de 1770, définissant le déplacement parallactique apparent du centre du disque de Vénus en vertu de la position topographique de la station d'observation par rapport au centre de la Terre. (Image: A Verdun)

in Latin may be translated as: «*Exposition of the methods for the determination of the solar parallax using observations of a transit of Venus as well as for finding the longitudes of places on Earth using observations of solar eclipses, along with the calculations and the conclusions*»[22]. At first sight it may astonish one to learn from this title that the geographic longitudes of observation stations are determined by means of solar eclipses. Normally, longitudes were determined using eclipses of the moons of Jupiter which happen far more frequent, or using occultations of stars by the Moon, or simply using lunar distances. There are, however, two reasons for this title. On the one hand transits actually are nothing else than partial solar eclipses and may thus be calculated principally by one and the same theory (if this is formulated generally enough). On the other hand only a few hours after the 1769 transit of Venus a total solar eclipse actually took place (see Table 4). This is why Euler formulated his model to such an extent of generality that he was able to process not only transit observations but even observations of the solar eclipse for improving the positions of those stations from where the eclipse was seen.

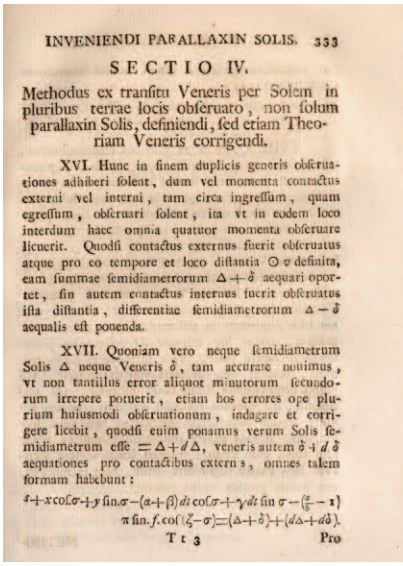
Euler's treatise may be summarized as follows. The advantage of his method consists in the way he formulated the observation equations, and in the fact that he extended them to *equations of condition* thus optimally adapting them to the special problem. He probably started from the idea that the angular distance between the centres of two point-like or extended celestial bodies being in conjunction is the crucial quantity for both theory and observation. Although this angular separation in the case of a transit of Venus could not be measured directly in those times, Euler introduced it as observable in his observation equations anyway. In Figures 15, 16, 17, and 18 the parameters and their meaning are illustrated from the original publication. Euler derived the observation equations in three steps:

Step 1: First he determines the geocentric angular distance ⊙♀ between the centre of the Sun's ⊙ and the centre of Venus' disk ♀ for the instant of their conjunction. May T be the epoch of conjunction of Sun and Venus given in mean time of Paris taken from astronomical tables. For this instant of time T the following elements may be given by the tables as well:

Ecliptic length of the Sun	=	L
Distance between Earth and Sun	=	a
Apparent radius of the Sun's disk	=	Δ
Hourly ecliptic motion of the Sun	=	α
Geocentric ecliptic length of Venus	=	L
Geocentric ecliptic latitude of Venus	=	l
Distance between Earth and Venus	=	b
Apparent radius of Venus' disk	=	δ
Hourly motion of Venus in ecliptic length	=	β
Hourly motion of Venus in ecliptic latitude	=	γ

The elements of the Sun resulting from the solar theory may be assumed accurate. For Venus, however, corrections (improvements) in length x and in latitude y have to be introduced so that the exact geocentric values for the ecliptic length will be given by $L + x$ and for the ecliptic latitude by $l + y$. For an arbitrary observation epoch $T + t$, where t is measured in hours before and after the instant of conjunction T, the following quantities may be defined:

[22] Expositio methodorum, cum pro determinanda parallaxi solis ex observato transitu Veneris per Solem, tum pro inveniendis longitudinibus locorum super terra, ex observationibus eclipsium solis, una cum calculis et conclusionibus inde deductis.



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SECTIO IV.

Methodus ex tranſitu Veneris per Solem in pluribus terrae locis obſeruato, non ſolum parallaxin Solis, definiendi, ſed etiam Theoriam Veneris corrigendi.

XVI. Hunc in finem duplicis generis obſeruationes adhiberi ſolent, dum vel momenta contactus externi vel interni, tam circa ingreſſum, quam egreſſum, obſeruari ſolent, ita vt in eodem loco interdum haec omnia quatuor momenta obſeruare licuerit. Quodſi contactus externus fuerit obſeruatus atque pro eo tempore et loco diſtantia $\odot v$ definita, eam ſummae ſemidiametrorum $\Delta + \delta$ aequari oportet, ſin autem contactus internus fuerit obſeruatus iſta diſtantia, differentiae ſemidiametrorum $\Delta - \delta$ aequalis eſt ponenda.

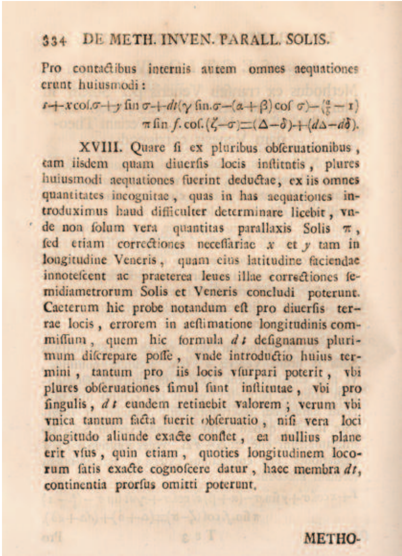
XVII. Quoniam vero neque ſemidiametrum Solis Δ neque Veneris δ, tam accurate nouimus, vt non tantillus error aliquot minutorum ſecundorum irrepere potuerit, etiam hos errores ope plurium huiusmodi obſeruationum, indagare et corrigere licebit, quodſi enim ponamus verum Solis ſemidiametrum eſſe $= \Delta + d\Delta$, veneris autem $\delta + d\delta$ aequationes pro contactibus externis, omnes talem formam habebunt:

$$s + x\,\mathrm{cof.}\sigma + y\,\mathrm{fin.}\sigma - (\alpha+\beta)\,dt\,\mathrm{cof.}\sigma + \gamma dt\,\mathrm{fin}\,\sigma - (\tfrac{a}{b} - 1)\,\pi\,\mathrm{fin.}f.\mathrm{cof}(\zeta-\sigma) = (\Delta+\delta) + (d\Delta + d\delta).$$

Tt 3 Pro

Figure 19: Page 333 from Euler's treatise of 1770, illustrating the observation equations formulated as equations of condition for the external contacts. (Image: A Verdun)

Figure 19: Page 333 du traité d'Euler de 1770, illustrant les équations d'observation formulées comme équations de conditions pour les contacts extérieurs. (Image: A Verdun)



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Pro contactibus internis autem omnes aequationes erunt huiusmodi:

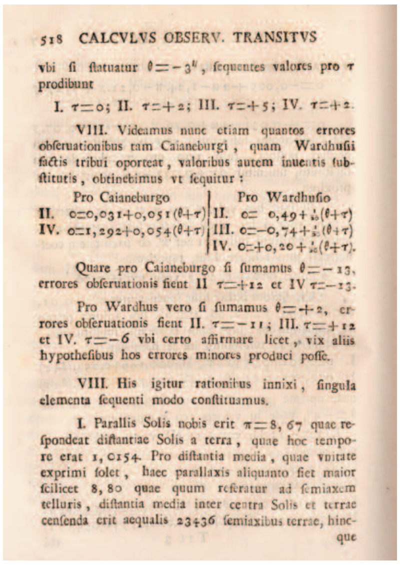
$$s + x\,\mathrm{cof.}\sigma + y\,\mathrm{fin}\,\sigma + dt(\gamma\,\mathrm{fin.}\sigma - (\alpha+\beta)\,\mathrm{cof}\,\sigma) - (\tfrac{a}{b} - 1)\,\pi\,\mathrm{fin}\,f.\mathrm{cof.}(\zeta-\sigma) = (\Delta-\delta) + (d\Delta - d\delta).$$

XVIII. Quare ſi ex pluribus obſeruationibus, tam iisdem quam diuerſis locis inſtitutis, plures huiusmodi aequationes fuerint deductae, ex iis omnes quantitates incognitae, quas in has aequationes introduximus haud difficulter determinare licebit, vnde non ſolum vera quantitas parallaxis Solis π, ſed etiam correctiones neceſſariae x et y tam in longitudine Veneris, quam eius latitudine faciendae innoteſcent ac praeterea leues illae correctiones ſemidiametrorum Solis et Veneris concludi poterunt. Caeterum hic probe notandum eſt pro diuerſis terrae locis, errorem in aeſtimatione longitudinis commiſſum, quem hic formula dt deſignamus plurimum diſcrepare poſſe, vnde introductio huius termini, tantum pro iis locis vſurpari poterit, vbi plures obſeruationes ſimul ſunt inſtitutae, vbi pro ſingulis, dt eundem retinebit valorem; verum vbi vnica tantum facta fuerit obſeruatio, niſi vera loci longitudo aliunde exacte conſtet, ea nullius plane erit vſus, quin etiam, quoties longitudinem locorum ſatis exacte cognoſcere datur, haec membra dt, continentia prorſus omitti poterunt.

METHO-

Figure 20: Page 334 from Euler's treatise of 1770, illustrating the observation equations formulated as equations of condition for the internal contacts. (Image: A Verdun)

Figure 20: Page 333 du traité d'Euler de 1770, illustrant les équations d'observation formulées comme équations de conditions pour les contacts intérieurs. (Image: A Verdun)



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vbi ſi ſtatuatur $\theta = -3''$, ſequentes valores pro τ prodibunt

I. $\tau = 0$; II. $\tau = +2$; III. $\tau = +5$; IV. $\tau = +2$.

VIII. Videamus nunc etiam quantos errores obſeruationibus tam Caianeburgi, quam Wardhuſii factis tribui oporteat, valoribus autem inuentis ſubſtitutis, obtinebimus vt ſequitur:

Pro Caianeburgo	Pro Wardhuſio
II. $0 = 0{,}031 + 0{,}051(\theta+\tau)$	II. $0 = 0{,}49 + \frac{1}{20}(\theta+\tau)$
IV. $0 = 1{,}292 + 0{,}054(\theta+\tau)$	III. $0 = -0{,}74 + \frac{1}{20}(\theta+\tau)$
	IV. $0 = +0{,}20 + \frac{1}{20}(\theta+\tau)$.

Quare pro Caianeburgo ſi ſumamus $\theta = -13$, errores obſeruationis fient II $\tau = +12$ et IV $\tau = -13$.

Pro Wardhus vero ſi ſumamus $\theta = +2$, errores obſeruationis fient II. $\tau = -11$; III. $\tau = +12$ et IV. $\tau = -6$ vbi certo affirmare licet, vix aliis hypotheſibus hos errores minores produci poſſe.

VIII. His igitur rationibus innixi, ſingula elementa ſequenti modo conſtituamus.

I. Parallis Solis nobis erit $\pi = 8{,}67$ quae reſpondeat diſtantiae Solis a terra, quae hoc tempore erat 1,0154. Pro diſtantia media, quae vnitate exprimi ſolet, haec parallaxis aliquanto fiet maior ſcilicet 8,80 quae quum referatur ad ſemiaxem telluris, diſtantia media inter centra Solis et terrae cenſenda erit aequalis 23436 ſemiaxibus terrae, hincque

Figure 21: Page 518 from Euler's treatise of 1770, showing his resulting value of 8.80'' for the solar parallax. (Image: A Verdun)

Figure 21: Page 518 du traité d'Euler de 1770, montrant sa valeur résultante de 8.80'' pour la parallaxe solaire. (Image: A Verdun)

Ecliptic length of the Sun	=	$L + \alpha t$
Geocentric ecliptic length of Venus	=	$L + \beta t + x$
Geocentric ecliptic latitude of Venus	=	$l + \gamma t + y$

The geocentric angular distance ⊙♀ between the centres of the Sun's ⊙ and Venus' disk ♀ may be calculated by the rectangular triangle ⊙♀V (Figure 17), where AB represents the ecliptic, ⊙ the centre of the Sun's disk, ♀ the centre of Venus' disk, and V the projection of ♀ to AB . Thus ⊙♀ $= s + x \cos\sigma + y \sin\sigma$, where s is an approximate value for ⊙♀ taken from the tables and σ is the angle ♀⊙V. Because of the fact that the hourly motions taken from the tables as well as the time measurements are subject to errors Euler introduces a time correction dt into the equation for ⊙♀, which has to be extended for $t + dt$:

$$\odot♀ = s + x\cos\sigma + y\sin\sigma - (\alpha+\beta)\,dt\cos\sigma + \gamma dt\sin\sigma.$$

<u>Step 2</u>: Now Euler reduces these elements to the pole of the equator and from there to the zenith of any place on Earth. The angle zR (Figure 18) is then given by

$$\mathrm{zR} = f - s\cos(\zeta - \sigma),$$

where z is the geocentric zenith, R is the geocentric position of ♀ which has been projected to the great circle z⊙, f is the angle ⊙z, and ζ is the angle z⊙B (Figure 18).

<u>Step 3</u>: Finally, Euler determines the apparent distance $\odot v$ between the centres of the Sun's and Venus' disks from the solar parallax π. The result is given approximately by

$$\odot v = s - ((a/b) - 1)\,\pi\sin f\cos(\zeta - \sigma).$$

The observation equation for $\odot v$ thus consists in four terms:

$$\odot v = s + x\cos\sigma + y\sin\sigma - (\alpha+\beta)\,dt\cos\sigma + \gamma\,dt\sin\sigma - ((a/b) - 1)\,\pi\sin f\cos(\zeta - \sigma).$$

The first term s represents the approximate value for the apparent angular separation $\odot v$ taken from the tables, which may be called *approximation term*. The second term $x\cos\sigma + y\sin\sigma$ contains the positioning errors introduced by the astronomical tables, which may be called *positioning term*. The third term

$$-(\alpha+\beta)\,dt\cos\sigma + \gamma\,dt\sin\sigma$$

contains the errors of the hourly motions introduced by the tables as well as the errors of the time measurements, which may be called *timing term*. The fourth and last term

$$-((a/b) - 1)\,\pi\sin f\cos(\zeta - \sigma)$$

contains the distances and the solar parallax, which may be called *distance* or *parallax term*.

It is worth noting that Euler's observation equations are formulated generally enough to process measurements of any angular distances between the centres of the Sun's and Venus' disks (i.e., not only those

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	Wardhus. Ingreſs. H. ′ ″	Wardhus. Egreſs. H. ′ ″	California. Ingreſs. H. ′ ″	California. Egreſs. H. ′ ″
Obſerved times.	9 34 10,6	15 27 24,6	0 17 27,9	5 54 50,3
Effect of parallax.	+ 6 35.6	— 4 35,9	+ 24.9	+ 4 52
Reduced times.	9 40 46,2	15 22 48,7	0 17 52,8	5 59 42.3

	Kola. Ingreſs.	Kola. Egreſs.	Hudſon's Bay. Ingreſs.	Hudſon's Bay. Egreſs.
Obſerved times.	9 42 4	15 35 23	1 15 23,3	7 0 47
Effect of parallax.	+ 6 37,4	— 4 45,1	+ 4 15,9	+ 0 38.7
Reduced times.	9 48 41,4	15 30 37,9	1 19 39,2	7 1 25,7

King George's Iſland.

	Ingreſs. Capt. Cook.	Ingreſs. Mean.	Ingreſs. Dr. Solander.	Ingreſs. Mr. Green.	Egreſs. Mean.
Obſerved times.	21 44 15,5	21 44 4,5	21 44 2,5	21 43 55,5	3 14 8
Effect of parallax.	5 40,4	5 40,4	5 40,4	5 40,4	+ 6 23,8
Reduced times.	21 38 35,1	21 38 24,1	21 38 22,1	21 38 15,1	3 20 31,8
Ditto at Wardhus.	9 40 46,2	9 40 46,2	9 40 46,2	9 40 46,2	15 22 48,7
Difference of meridians.	12 2 11,1	12 2 22,1	12 2 24.1	12 2 31,1	12 2 16,9

	Ingreſs. Capt. Cook.	Ingreſs. Mean.	Ingreſs. Dr. Solander.	Ingreſs. Mr. Green.	Egreſs. Mean.
Reduced times at K. G. Iſl.	21 38 35,1	21 38 24,1	21 38 22,1	21 38 15,1	3 20 31,8
Ditto at California.	0 17 52,8	0 17 52,8	0 17 52,8	0 17 52,8	5 59 42,3
Difference of meridians.	2 39 17.7	2 39 28,7	2 39 30,7	2 39 37,7	2 39 10,5

	Ingreſs. Capt. Cook.	Ingreſs. Mean.	Ingreſs. Dr. Solander.	Ingreſs. Mr. Green.	Egreſs. Mean.
Reduced times at K. G. Iſl.	21 38 35,1	21 38 24,1	21 38 22,1	22 38 15,1	3 20 31,8
Ditto at Kola.	9 48 41,4	9 48 41,4	9 48 41,4	9 48 41,4	15 30 37,9
Difference of meridians.	12 10 6,3	12 10 17,3	12 10 19,3	10 10 26,3	12 10 6,1

	Ingreſs. Capt. Cook.	Ingreſs. Mean.	Ingreſs. Dr. Solander.	Ingreſs. Mr. Green.	Egreſs. Mean.
Reduced times at K. G. Iſl.	21 38 35,1	21 38 24,1	21 38 22,1	21 38 15,1	3 20 31,8
Ditto at Hudſon's Bay.	1 19 39,2	1 19 32,9	1 19 39,2	1 19 39.2	7 1 25,7
Difference of meridians.	3 41 4,1	3 41 15,1	3 41 17,1	3 41 24,1	3 40 53,9

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Figure 22: Page 577 from the treatise published by Hornsby in 1772, showing the «effect of parallax». (Image: A. Verdun)

Figure 22: Page 577 du traité publié par Hornsby en 1772, montrant «l'effet de parallaxe». (Image: A. Verdun)

associated with the instants of the internal contacts or the smallest angular separation). Because it was not possible from the technical point of view to perform such kinds of observations in the 18th century, Euler had to adopt his observation equations to the measured instants of contact. For this reason he set up the following equations of condition for the external and internal contacts:

for the external contacts $\odot v = (\Delta + \delta) + (d\Delta + d\delta)$
for the internal contacts $\odot v = (\Delta - \delta) + (d\Delta - d\delta)$,

where $d\Delta$ and $d\delta$ represent the «uncertainties» of the apparent radii of the Sun's and Venus' disks. These corrections are introduced as parameters which have to be estimated as well.

From a series of such equations derived from observations performed at one and the same or at different observation stations all unknowns may be estimated, particularly the parameters π, x, y, and dt. Euler's observation equations are illustrated in Figures 19 and 20. Using these equations and the observations of the solar eclipse he first determined precise values for the longitudes of some observation stations. Then he processed the observation data acquired from the 1769 transit of Venus. Only the most important steps of his parameter estimation method are briefly mentioned, because the calculations in his treatise cover over 130 pages:

1. Elimination of parameters by appropriate combinations of the equations of condition leaving only the parameters x, y, and π in the observation equations.
2. Grouping the equations of condition into four classes according to the instants of contact.
3. Setting up mean equations of condition per class (by averaging the coefficients of the equations).
4. Determination of first approximation values of all remaining parameters by appropriate combinations of the mean equations of condition.
5. Improvement of the astronomical elements resp. of the theoretical a priori parameters resulting from them.
6. Setting up new equations of condition containing correction terms using the improved elements.
7. Setting up error equations for the observations containing the corrections as unknowns.
8. Determination of the corrections in such a way that the observation errors will become minimal and will assume positive as well as negative values.

Euler's result for the mean solar parallax is shown in Figure 21. His value of $\pi_{\odot}$ = 8.80" is close to the present value. In an appendix to his treatise Euler confirmed this result by processing the observations acquired in California. Whether this excellent result was realized merely by chance or by Euler's adjustment procedure, which was performed not without some arbitrariness, may be judged only by reprocessing just the same observations as available to Euler using a modern parameter estimation method based on least squares adjustment. It may, in fact, be expected to deliver the same result, although Euler's parameter estimation is not perfect from the modern point of view. His goals (minimizing the residuals, no systematic errors), however, correspond clearly to modern scientific requirements.

Worthy of mention is a small but interesting detail in Euler's treatise. The instants of conjunction for the solar eclipse and for the transit of Venus given by Euler are June 3, 1769, 20^h 30^m 26^s and June 3, 1769, 10^h 7^m 39^s, respectively, both in mean time for the meridian of Paris. Considering the time difference between Paris and Greenwich being 9^m 19^s which has to be added to the epochs as given by Euler to get them in Universal Time (UT), yields June 3, 1769, 20^h 39^m 45^s and June 3, 1769, 10^h 16^m 58^s, respectively. According to Espenak[23] these epochs are June 4, 1769, 08^h 28^m and June 3, 1769, 22^h 25^m. These epochs coincide with those given by Euler only if 12 hours are added to Euler's epochs, which

[23] Cf. http://sunearth.gsfc.nasa.gov/eclipse/transit/ catalog/VenusCatalog.html and http://sunearth.gsfc.nasa.gov/eclipse/solar.html

means that for Euler the day starts at noon as it is commonly used in Astronomy today (Julian date). This fact raises the question since when are hours of epochs (i.e., fractions of the days) actually counted from noon in astronomy. Since the introduction of the Julian date or Julian day numbers? But this probably became common use and standard only during the 19th century. Moreover, this comparison between the epochs given by Euler and Espenak shows that the instants of conjunction, which Euler may have extracted from the best astronomical tables then available, differ by about 10 minutes. What is the reason for this difference? Is this difference caused by the value of the solar parallax then used to construct the astronomical tables? Let the answer be subject to further research and let us focus on the determination of the solar parallax by other scientists of the 18th and 19th century.

Before continuing with the discussion of the treatise written by Duséjour the results achieved by Euler's contemporaries, Hornsby and Pingré, have to be inspected briefly.

Hornsby did not change the well established method of averaging in his treatise[24] of 1772. It is striking, however, that now he uses the value π_{Theory} = 8.7" for the a priori solar parallax. He compares the transit durations Δt_{32} measured at 5 stations with one another and achieved the result π_{Obs} = 8.65" using the formula $\pi_{Obs} = (\Delta_{Obs} / \Delta_{Theory})\, \pi_{Theory}$, which yields $\pi_{\odot}$ = 8.78". He seemed at least to have recognized that the business may be turned around to see the effect of π_{Obs} = 8.65" on the meridian differences if assuming this value be correct, reducing the observations to certain meridians and calculating the meridian differences (Figure 22). He thus analyzed the (indirect) «effect of parallax» on the observations. The next step would have consisted in the realization that one has to vary the parameter to be estimated in such a way that the «effect of parallax» on the differences of the reduced observations, which were calculated with this parameter, will be as small as possible.

With no doubt Pingré has stolen the show with his treatise[25] published in 1775. In the introduction he wrote: «*Je me crois en état de prouver, j'oserois presque dire de démontrer rigoureusement, ou que cette parallaxe est à peu-près telle que Mrs. Euler & Hornsby l'ont déterminée, ou qu'on ne peut rien conclure de la durée du dernier pas-*

24 Cf. Hornsby (1772).

25 Cf. Pingré (1775).

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TABLE des effets de la parallaxe de Vénus au Soleil ſur les inſtans des principales phaſes du paſſage de Vénus, la parallaxe moyenne du Soleil étant ſuppoſée de 8",5.

NOMS des LIEUX.	EFFET DE LA PARALLAXE SUR LE 2.e Contact. M. S.	3.e Contact. M. S.	4.e Contact. M. S.	LATITUDE des LIEUX. D. M. S.	LEUR LONGITUDE. H. M. S.
Taïti	+5. 30,3	−6. 12,6	−6. 7,7	17. 28. 55 S.	10. 7. 19 O.
Saint-Joſeph	−0. 18,6	−4. 42,9	−4. 40,8	23. 3. 42 N.	7. 28. 10
Hudſon	−4. 8,4	−0. 37,0	−0. 48,2	58. 47. 32	6. 26. 16
Iſle Coudre	−4. 38,5	......	......	47. 31. 41	4. 50. 18
New Cambridge	−4. 17,5	......	......	42. 25. 0	4. 54. 0
Norriton	−3. 55,1	......	......	40. 9. 56	5. 11. 35
Philadelphie	−3. 54,2	......	......	39. 56. 55	5. 10. 24
Leweſtown	−3. 50,1	......	......	38. 47. 27	5. 9. 45 ½
Mexico	−0. 38,5	......	......	19. 54. 0	6. 49. 43
Le Cap françois	−2. 34,9	......	......	19. 47. 4	4. 58. 32
La Martinique	−2. 55,9	......	......	14. 44. 0	4. 8. 58 ::
Cadiz	−6. 40,5	......	......	36. 31. 7	0. 34. 37
Greenwich	−6. 56,9	......	......	51. 28. 40	0. 9. 16
Hawkhill	−6. 55,6	......	......	55. 57. 37	0. 21. 56
Paris	−6. 57,7	......	......	48. 50. 14	0. 0. 0
Upſal	−6. 50,0	......	......	59. 51. 50	1. 1. 13 E.
Stockolm	−6. 50,5	......	......	59. 20. 31	1. 2. 53
Hernoſand	−6. 48,5	......	......	60. 38. 0	1. 2. 12
Cajanebourg	−6. 38,0	+4. 50,0	+4. 30,8	64. 13. 30	1. 41. 41
Saint-Péterſbourg	......	+5. 16,5	+4. 51,7	59. 56. 23	1. 52. 0
Wardhus	−6. 24,2	+4. 28,1	+4. 6,8	70. 22. 36	1. 5 7 ::
Kola	−6. 26,0	+4. 36,9	+4. 15,9	68. 52. 55	2. 2. 43
Ponoi	−6. 21,8	......	......	67. 4. 30	2. 35. 11 ::
Gurief	......	+6. 19,7	+5. 58,1	47. 7. 8	3. 18. 28
Orembourg	......	+6. 5,7	+5. 42,7	51. 46. 0	3. 31. 0
Orsk	......	+6. 11,4	......	51. 12. 11	3. 44. 28 ::
Batavia	......	+3. 56,5	+3. 35,6	6. 12. 0 S.	6. 58. 15
Pékin	......	+4. 24,2	+3. 51,0	39. 55. 15 N	7. 36. 23
Manille	......	+3. 36,8	+3. 8,2	14. 36. 8	7. 54. 4 ½

Mém. 1772. Fff

Figure 23: Page 409 from the treatise published by Pingré in 1775, showing the «effets de la parallaxe». (Image: A. Verdun)

Figure 23: Page 409 du traité publié par Pingré en 1775, montrant les «effets de la parallaxe». (Image: A. Verdun)

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ſortie plus de durée qu'elle ne devoit en avoir, il eſt probable qu'ils ſe ſont trompés dans l'obſervation du ſecond contact intérieur. Dans le calcul de la durée, on a ſuppoſé la parallaxe du Soleil, de 8",5 ; ſi on eût ſuppoſé 8",8, la durée calculée auroit été un peu plus longue à Taïti ; preſque la même à Saint-Joſeph, plus courte par-tout ailleurs. On peut encore conclure, ſi l'on veut, de cette Table, que la méthode de déterminer le diamètre de Vénus par la durée de ſa ſortie, quoiqu'excellente dans la théorie, eſt incertaine dans la pratique, & que par conſéquent des combinaiſons fondées ſur la connoiſſance préciſe de ce diamètre ne peuvent donner un réſultat bien certain.

TABLE des durées calculées & obſervées de la ſortie de Vénus.

NOMS des OBSERVATEURS.	LIEUX des OBSERVATIONS.	DURÉE calculée.	DURÉE obſervée.	ERREUR de l'Obſervat.
Le P. Hell	Wardhus	18′ 20″,4	18′ 18″,8	— 1″,6
M. Mohr	Batavia	18. 20,8	18. 18,0	— 2,8
Le P. Collas	Pékin	18. 8,5	18. 5,0	— 3,5
M. Lowitz	Gurief	18. 20.1	18. 11,0	— 9,1
M. Borgrewing	Wardhus	18. 20,4	18. 9,8	— 10,6
Le P. Sajnovics	Wardhus	18. 20,4	18. 8,8	— 11,6
L'Abbé Chappe	Saint-Joſeph	18. 43,8	18. 28,8	— 15,0
M. Wales	Baie d'Hudſon	18. 30,5	18. 15,7	— 15,0
Le P. Mayer	Péterſbourg	18. 22,9	17. 56,3	— 26,6
M. de Ronas	Manille	18. 13,1	17. 41,0	— 32,1
M. Green	Taïti	18. 46,6	18. 11,0	— 35,6
M. Euler, M. Lexell	Péterſbourg	18. 22,9	17. 43,0	— 39,9
Le P. Stahl	Péterſbourg	18. 22,9	17. 40,0	— 42,9
M. Médina	Saint-Joſeph	18. 43,8	17. 58,5	— 45,3
M. Doz	Saint-Joſeph	18. 43,8	17. 53,5	— 50,3
M. Cook	Taïti	18. 46,6	17. 49,0	— 57,6
M. Dymond	Baie d'Hudſon	18. 30,5	18. 31,7	+ 1,2
M. Krafft	Orenburg	18. 18,7	18. 28,0	+ 9,3
M. Ch. Euler	Orsk	18. 19,0	18. 31,0	+ 12,0
Le P. Dollieres	Pékin	18. 8,5	18. 36,0	+ 27,5

Figure 24: Page 420 from the treatise published by Pingré in 1775, showing the «observation errors». (Image: A. Verdun)

Figure 24: Page 420 du traité publié par Pingré en 1775, montrant les «erreurs d'observation». (Image: A. Verdun)

sage de Vénus.» In his earlier publications he supposed the a priori value for the solar parallax to be 10", in this treatise, however, he used $\pi_{\text{Theory}} = 8.80"$. He compares the transit durations Δt_{32} measured at 5 stations with the corresponding values measured at 5 other stations and obtained the result $\pi_{\text{Obs}} = 8.78"$. He concludes that the solar parallax has to be $\pi_{\odot} = 8.8"$, supposing to have proved his introducing statement. Apart from this rather doubtful argumentation even Pingré seemed to have realized (Figures 23 and 24), that different values for the parallaxes also have a different «effet de la parallaxe» on the reduced quantities to be compared with or produce a different «erreur de l'observation».

Finally, the very remarkable treatise[26] written by Achille-Pierre Dionis Duséjour (or Du Séjour) (1734-1794) is presented. It is the sixteenth Mémoire out of a series consisting of 18 Mémoires published by Duséjour between 1767 and 1786 in the *Histoire de l'Académie Royale des Sciences avec les Mémoires de Mathématique et de Physique, Tirés des Registres de cette Académie* for the years 1764 until 1783. These Mémoires contain more than 2000 pages and are devoted to the determination of eclipses and lunar occultations as well as to the processing and reduction of astronomical observations. Duséjour published these treatises some time later in his two-volume textbook[27]. It is rather strange that his works obviously were almost totally ignored by the scientific community; perhaps because he was not a professional astronomer[28]. There is only one exception. The astronomer Jean-Baptiste-Joseph Delambre (1749-1822), who was known by his theoretical and historical contributions to astronomy and who published together with Pierre-François-André Méchain (1744-1804) the fundamental work *Base du système métrique décimal*[29] (introducing the decimal system officially by this work) which made both authors famous world-wide, devoted 27 pages to Duséjour's work in his *Histoire de l'Astronomie au Dix-Huitième Siècle*[30], thus revealing his great and honest respect for Duséjour's achievements. In the *Dictionary of Scientific Biography* René Taton wrote about Duséjour's work[31]: «*All these works are dominated by an obvious concern for rigor and by a great familiarity with analytical methods; if the prolixity of the developments and the complexity of the calculations rendered them of little use at the time, their reexamination in the light of present possibilities of calculation would certainly be fruitful*». Another reason for «disregarding» Duséjour's work may be found in his extremely compact style of writing. In the sixteenth Mémoire mentioned above he used symbols again and again which were defined elsewhere in his previous treatises (which nevertheless contain about 1800 pages). An inventory of the definitions of the symbols, parameters and concepts relevant for this treatise may be ferreted out, e.g., in the eighth Mémoire published in 1773. Two pages of this list, which counts several pages, are illustrated in Figures 25 and 26. Let us start now with the discussion of his processing method.

Be Z' the point of reference (e.g., the Earth's centre) and Z the hour angle of Z' at the instant of conjunction, given in units of time. Be z' the position of an observation station and z the hour angle of z' at the observation epoch, also given in units of time. The longitude y resp. Y, which (apart from transformation terms) essentially is defined by the difference between the hour angles z and Z, have to be determined considering whether the hour angles at the observation epochs have to be measured to the east or west of the reference meridian. For keeping the matter as simple as possible only the quantities y and it's derivative dy are considered. The observables, i.e., the instants of contact measured at an observation station, occur as time arguments (observation epochs) in the model for y which contains all relevant parameters. In particular, y depends on the distance between the centres of the Sun's and Venus' disks. The «correction» dy depends on the derivatives of y with respect to the model parameters, represented by the coefficients of the «correction terms». The goal is to determine these correction terms associated with the various parameters from the contact observations using equations of condition defined by the durations Δt_{32} and Δt_{42} of the transit. The elements provided by the astronomical tables and needed as initial values for the model are shown in Figure 27 for the 1761 transit. Note the a priori values for the solar parallax of 8.60" (for the 1761 transit) and 8.62" (for the 1769 transit) which correspond to the epochs of the respective transits.

In a next step $y + dy$ is calculated for each of the two transits, for each observation station, and for each instant of contact (Figure 28). Then two types of equations of condition per station and transit are set up:

$$\text{Type 1 (for } \Delta t_{42}\text{):}\quad y'' - y + dy'' - dy = 0$$
$$\text{Type 2 (for } \Delta t_{32}\text{):}\quad y' - y + dy' - dy = 0,$$

where y and dy concern the instant of the second contact, y' and dy' of the third contact, and y'' and dy'' of the fourth contact. These equations of condition are functions of the corrections (improve-

[26] Cf. Duséjour (1784).
[27] Cf. Duséjour (1786).
[28] Duséjour was a politician and a member of the Parliament.
[29] Cf. Méchain and Delambre (1806).
[30] Cf. Delambre (1827).
[31] Cf. Taton (1971).

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ont un ſigne contraire avec le même coëfficient numérique; on donneroit donc également l'explication des phénomènes, en admettant une diminution réelle dans le demi-diamètre de la Lune.

RÉCAPITULATION des Sections précédentes.

Équation complète aux Longitudes.

(122.) J'ai épuiſé, ce me ſemble, toutes les façons poſſibles de faire varier les équations du *§. 65;* je puis donc déterminer maintenant l'équation complète aux Longitudes, en lui donnant la forme la plus générale dont elle ſoit ſuſceptible.

Soit

Z' le lieu d'où l'on compte les Longitudes.

Z l'angle horaire du lieu Z' à l'inſtant de la conjonction. Je ſuppoſe cet angle évalué en temps.

z' le lieu où l'on a obſervé, & dont on cherche la différence en longitude avec le lieu Z'.

z l'angle horaire du lieu z' à l'inſtant de l'obſervation. Je ſuppoſe cet angle évalué en temps.

b' le nombre de ſecondes horaires écoulées, depuis l'inſtant de la conjonction donné par les Tables aſtronomiques juſqu'à l'inſtant de l'obſervation, ou calculé par la formule du *§. 69.*

$$A = \frac{\psi l}{\zeta} - \frac{q s \varphi}{r^2} + \frac{c g \rho \omega}{r^3} + \frac{c h p \rho \varphi}{r^4};$$

$$F = \frac{\theta l}{\zeta} - \frac{q s \omega}{r^2} - \frac{c g \rho \varphi}{r^3} + \frac{c h p \rho \omega}{r^4};$$

$$E = \xi - \frac{p s \pi}{r^2} - \frac{c \rho q h \pi}{r^4} - \frac{b'^2 \gamma \pi}{3600''^2 r}.$$

$$L = \frac{\sigma \tau' r E}{\pi \zeta \partial'} - \frac{\delta \tau r}{\pi \zeta}$$ s'il s'agit d'un contact intérieur des limbes.

$$L = \frac{\sigma \tau' r E}{\pi \zeta \partial} + \frac{\delta \tau r}{\pi \zeta}$$ s'il s'agit d'un contact extérieur des limbes.

$$L = \frac{\lambda r E}{\pi \zeta}$$ s'il s'agit d'une diſtance quelconque des centres.

$N =$

Figure 25: Page 344 of the eighth Mémoire published by Duséjour in 1784, illustrating the definitions of the parameters. (Image: A. Verdun)

Figure 25: Page 344 du huitième Mémoire publié par Duséjour en 1784, illustrant les définitions des paramètres. (Image: A. Verdun)

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$$\mu = + \frac{3600'' \zeta}{206265'' n} \times A - \frac{3600'' \zeta}{206265'' n} \times \frac{A F}{\sqrt{(L^2 - A^2)}}.$$

$$\Xi = - \frac{3600'' r^3 \zeta}{206265'' n^2 \psi \pi} \times F + \frac{3600'' r^3 \zeta}{206265'' n^2 \psi \pi} \times \sqrt{(L^2 - A^2)}.$$

$$\Gamma = \begin{cases} + \dfrac{3600'' \zeta r}{206265'' n \pi} \times \left(F - \dfrac{\theta l}{\zeta}\right) - \dfrac{3600'' \zeta r}{206265'' n \pi} \times \sqrt{(L^2 - A^2)} \\ + \dfrac{3600'' \zeta r}{206265'' n \pi} \times \dfrac{L^2}{\sqrt{(L^2 - A^2)}} - \dfrac{3600'' \zeta r}{206265'' n \pi} \times \dfrac{\psi l}{\zeta} \dfrac{A}{\sqrt{(L^2 - A^2)}}. \end{cases}$$

$$\Theta = \frac{3600'' \theta r^2}{206265'' n \pi} + \frac{3600'' \psi r^2}{206265'' n \pi} \times \frac{A}{\sqrt{(L^2 - A^2)}}.$$

$$\Sigma = - \frac{3600'' r^2}{206265'' n \pi} \times \frac{L E}{\sqrt{(L^2 - A^2)}}.$$

$$y = z - Z + \frac{3600'' \zeta}{n r} \times F - \frac{3600'' \zeta}{n r} \sqrt{(L^2 - A^2)}.$$

$$dy = \begin{cases} + \dfrac{b' \zeta}{206265'' n \pi} \times \dfrac{L^2}{\sqrt{(L^2 - A^2)}} \text{ var. hor. (parall. horiz. pol. de la Lune)} \\ + \dfrac{b'}{3600''} \times \dfrac{\varepsilon}{r} \text{ var. hor. (déclin. du Soleil)} + \dfrac{\varepsilon}{r} d \text{(déclin. du Soleil)} \\ + \dfrac{\Delta}{r} d \text{ (inſtant de l'obſerv.)} + \dfrac{\beta}{r} d \text{ (latitude de l'Obſervatoire)} \\ + \varpi\, d \text{ (demi-grand axe terreſtre)} + \dfrac{\mu}{r} d \text{ (inclin. de l'orbite corr.)} \\ + \dfrac{\Xi}{r} d \text{ (mouv. hor. de la ☾ au ☉)} + \dfrac{\Gamma}{r} d \text{ (parall. horiz. pol. de la ☾)} \\ - \dfrac{\Gamma}{r} d \text{ (parall. horiz. du Soleil)} + \dfrac{\Theta}{r} d \text{ (latitude de la Lune.)} \\ \textit{Contacts intérieurs des limbes.} \\ + \dfrac{\Sigma}{r} d \text{ (demi-diam. du ☉)} + \dfrac{\Sigma}{r} d \text{ (inflexion)} - \dfrac{\Sigma}{r} d \text{ (demi-diam. de la ☾)}; \\ \textit{Contacts extérieurs des limbes.} \\ + \dfrac{\Sigma}{r} d \text{ (demi-diam. du ☉)} - \dfrac{\Sigma}{r} d \text{ (inflexion)} + \dfrac{}{r} d \text{ (demi-diam. de la ☾)}; \\ \textit{Diſtance quelconque des centres.} \\ + \dfrac{\Sigma}{r} d \text{ (diſtance des centres).} \end{cases}$$

On aura

Longitude du lieu z' — longitude du lieu Z' $\begin{cases} = Y + dY. \\ = y + dy. \end{cases}$

Figure 26: Page 348 of the eighth Mémoire published by Duséjour in 1784, continuing the definition of the parameters. (Image: A. Verdun)

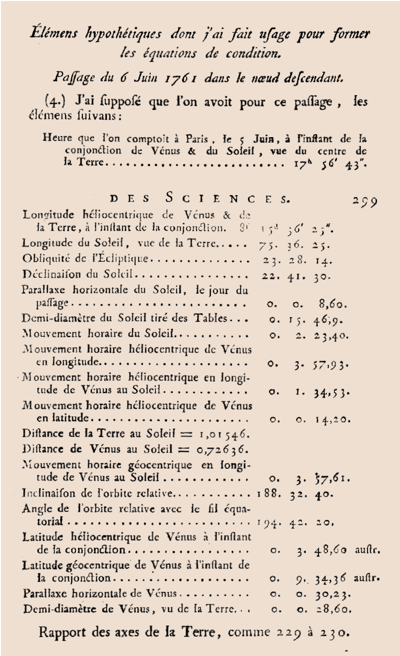
Figure 26: Page 348 du huitième Mémoire publié par Duséjour en 1784, poursuivant la définition des paramètres. (Image: A. Verdun)

ments) to the Sun's apparent radius, to the apparent geocentric latitude of Venus, to the solar parallax, and to the apparent geocentric hourly motion of Venus. Then all equations of condition are summed up per type and transit yielding four equations:

Equation I	for type 1 and for the 1761 transit
Equation II	for type 1 and for the 1769 transit
Equation III	for type 2 and for the 1761 transit
Equation IV	for type 2 and for the 1769 transit

Now these equations are solved (by combination and elimination procedures) for the corrections to the solar parallax, the geocentric latitude of Venus, and the apparent radius of Venus' disk, yielding two by two equations of condition per transit as well as one equation for the radius of Venus' disk, all these equations being functions of the corrections of the radius of the Sun's disk, of the hourly motion of Venus, and of the observations. The sum of the observation errors is assumed to be zero, which means that the errors in the differences of the measured instants of contact are statistically averaged out. The result is shown in Figures 29 and 30. Duséjour obtains (from both transits) for the value of the mean solar parallax $\pi_\odot = 8.8418''$. The reprocessing performed in his textbook yields the value $\pi_\odot = 8.851''$.

It is highly recommended to read and study Duséjour's method of data processing in the original publications, which was presented here only very briefly. Except for the treatise by Euler, it may be difficult to find any other parameter estimation published in the 1770ies, or earlier, written with similar rigour as by these two authors. It remains an open question, however, to what extent and in which respect their work had any influence on the development of the parameter estimation methods. It was claimed and is still claimed again and again, that the 18th century transits of Venus were a failure from the



Élémens hypothétiques dont j'ai fait uſage pour former les équations de condition.

Paſſage du 6 Juin 1761 dans le nœud deſcendant.

(4.) J'ai ſuppoſé que l'on avoit pour ce paſſage, les élémens ſuivans:

Heure que l'on comptoit à Paris, le 5 Juin, à l'inſtant de la conjonction de Vénus & du Soleil, vue du centre de la Terre........................ 17h 56′ 43″.

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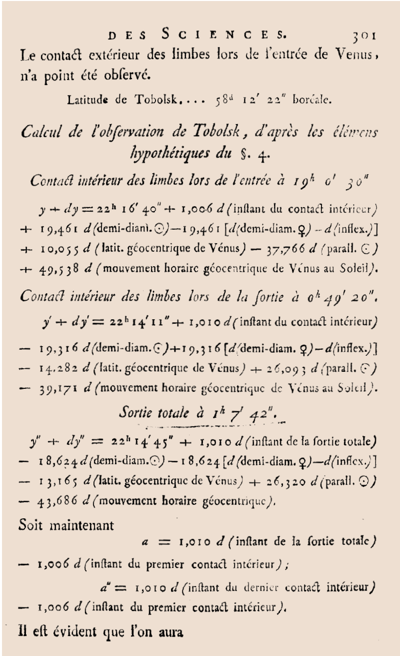
Longitude héliocentrique de Vénus & de la Terre, à l'inſtant de la conjonction.	8s 15d	56′	25″.
Longitude du Soleil, vue de la Terre.....	75.	36.	25.
Obliquité de l'Écliptique..............	23.	28.	14.
Déclinaiſon du Soleil................	22.	41.	30.
Parallaxe horizontale du Soleil, le jour du paſſage..........................	0.	0.	8,60.
Demi-diamètre du Soleil tiré des Tables...	0.	15.	46,9.
Mouvement horaire du Soleil...........	0.	2.	23,40.
Mouvement horaire héliocentrique de Vénus en longitude.....................	0.	3.	57,93.
Mouvement horaire héliocentrique en longitude de Vénus au Soleil............	0.	1.	34,53.
Mouvement horaire héliocentrique de Vénus en latitude......................	0.	0.	14,20.
Diſtance de la Terre au Soleil = 1,01546.			
Diſtance de Vénus au Soleil = 0,72636.			
Mouvement horaire géocentrique en longitude de Vénus au Soleil............	0.	3.	37,61.
Inclinaiſon de l'orbite relative............	188.	32.	40.
Angle de l'orbite relative avec le fil équatorial...........................	194.	42.	20.
Latitude héliocentrique de Vénus à l'inſtant de la conjonction.................	0.	3.	48,60 auſtr.
Latitude géocentrique de Vénus à l'inſtant de la conjonction..................	0.	9.	34,36 auſtr.
Parallaxe horizontale de Vénus..........	0.	0.	30,23.
Demi-diamètre de Vénus, vu de la Terre...	0.	0.	28,60.

Rapport des axes de la Terre, comme 229 à 230.

Figure 27: Parts of the pages 298 and 299 of the treatise published by Duséjour in 1784, illustrating the approximation values of the elements for the 1761 transit taken from astronomical tables.
(Image: A. Verdun)
Figure 27: Extraits des pages 298 et 299 du traité publié par Duséjour en 1784, illustrant les valeurs d'approximation des éléments pour le transit de 1761 tirés des tables astronomiques.
(Image: A. Verdun)



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Le contact extérieur des limbes lors de l'entrée de Venus, n'a point été obſervé.

Latitude de Tobolsk.... 58d 12′ 22″ boréale.

Calcul de l'obſervation de Tobolsk, d'après les élémens hypothétiques du §. 4.

Contact intérieur des limbes lors de l'entrée à 19h 0′ 30″

$y + dy$ = 22h 16′ 40″ + 1,006 d (inſtant du contact intérieur)
+ 19,461 d(demi-diam.☉) − 19,461 [d(demi-diam.♀) − d(inflex.)]
+ 10,055 d (latit. géocentrique de Vénus) − 37,766 d (parall. ☉)
+ 49,538 d (mouvement horaire géocentrique de Vénus au Soleil).

Contact intérieur des limbes lors de la ſortie à 0h 49′ 20″.

$y' + dy'$ = 22h 14′ 11″ + 1,010 d(inſtant du contact intérieur)
− 19,316 d(demi-diam.☉) + 19,316 [d(demi-diam. ♀) − d(inflex.)]
− 14,282 d (latit. géocentrique de Vénus) + 26,093 d (parall. ☉)
− 39,171 d (mouvement horaire géocentrique de Vénus au Soleil).

Sortie totale à 1h 7′ 42″.

$y'' + dy''$ = 22h 14′ 45″ + 1,010 d(inſtant de la ſortie totale)
− 18,624 d(demi-diam.☉) − 18,624 [d(demi-diam. ♀) − d(inflex.)]
− 13,165 d(latit. géocentrique de Vénus) + 26,320 d(parall. ☉)
− 43,686 d (mouvement horaire géocentrique).

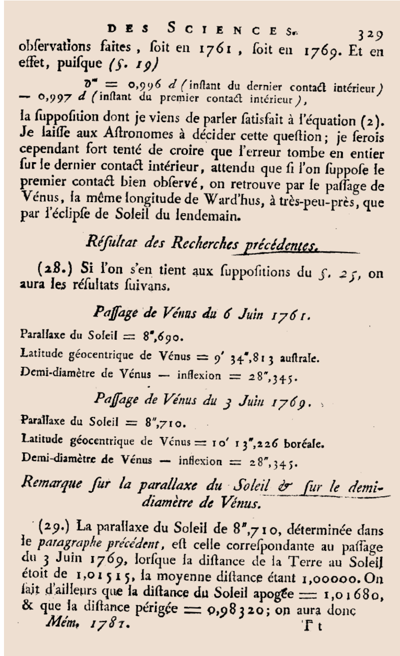
Soit maintenant

a = 1,010 d (inſtant de la ſortie totale)
− 1,006 d (inſtant du premier contact intérieur);

a'' = 1,010 d (inſtant du dernier contact intérieur)
− 1,006 d (inſtant du premier contact intérieur).

Il eſt évident que l'on aura

Figure 28: Page 301 of the treatise published by Duséjour in 1784, showing the resulting equations of condition for $y + dy$ concerning the instant of the second contact, $y' + dy'$ of the third contact, and $y'' + dy''$ of the fourth contact.
(Image: A. Verdun)
Figure 28: Page 301 du traité publié par Duséjour en 1784, montrant les équations de conditions pour $y + dy$ concernant l'instant du second contact, $y' + dy'$ du troisième contact, et $y'' + dy''$ du quatrième contact.
(Image: A. Verdun)



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obſervations faites, ſoit en 1761, ſoit en 1769. Et en effet, puiſque (§. 19)

∂''' = 0,996 d (inſtant du dernier contact intérieur)
− 0,997 d (inſtant du premier contact intérieur),

la ſuppoſition dont je viens de parler ſatisfait à l'équation (2). Je laiſſe aux Aſtronomes à décider cette queſtion; je ſerois cependant fort tenté de croire que l'erreur tombe en entier ſur le dernier contact intérieur, attendu que ſi l'on ſuppoſe le premier contact bien obſervé, on retrouve par le paſſage de Vénus, la même longitude de Ward'hus, à très-peu-près, que par l'éclipſe de Soleil du lendemain.

Réſultat des Recherches précédentes.

(28.) Si l'on s'en tient aux ſuppoſitions du §. 25, on aura les réſultats ſuivans.

Paſſage de Vénus du 6 Juin 1761.

Parallaxe du Soleil = 8″,690.
Latitude géocentrique de Vénus = 9′ 34″,813 auſtrale.
Demi-diamètre de Vénus − inflexion = 28″,345.

Paſſage de Vénus du 3 Juin 1769.

Parallaxe du Soleil = 8″,710.
Latitude géocentrique de Vénus = 10′ 13″,226 boréale.
Demi-diamètre de Vénus − inflexion = 28″,345.

Remarque ſur la parallaxe du Soleil & ſur le demi-diamètre de Vénus.

(29.) La parallaxe du Soleil de 8″,710, déterminée dans le *paragraphe précédent*, eſt celle correſpondante au paſſage du 3 Juin 1769, lorſque la diſtance de la Terre au Soleil étoit de 1,01515, la moyenne diſtance étant 1,00000. On ſait d'ailleurs que la diſtance du Soleil apogée = 1,01680, & que la diſtance périgée = 0,98320; on aura donc

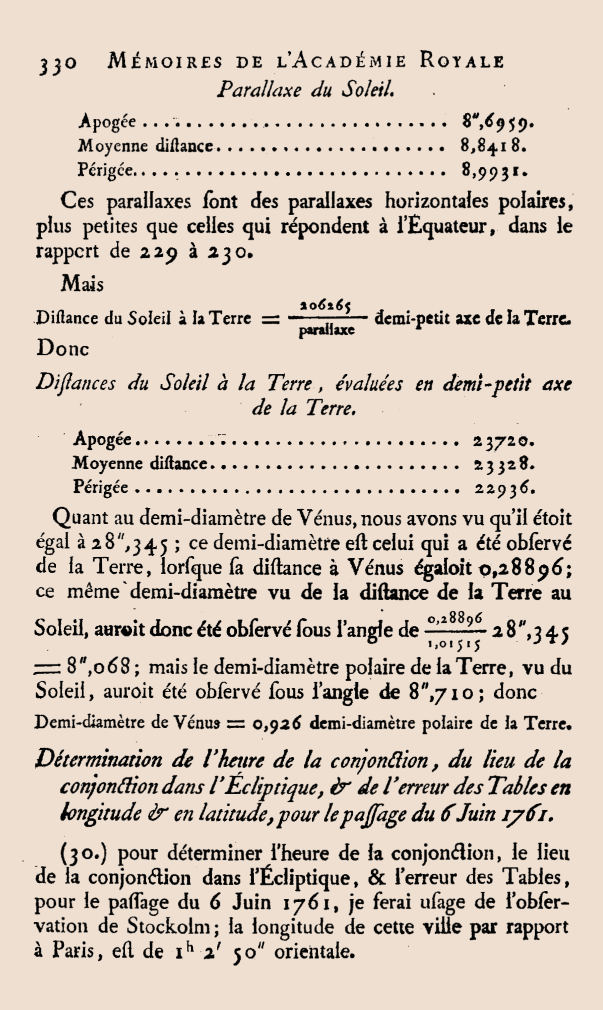
Mém. 1781. Tt

Figure 29: Page 329 of the treatise published by Duséjour in 1784, showing the results for the epochs of the 1761 and 1769 transits of Venus.
(Image: A. Verdun)
Figure 29: Page 329 du traité publié par Duséjour en 1784, montrant les résultants pour les époques des transits de Vénus de 1761 et de 1769.
(Image: A. Verdun)

scientific point of view due to the prejudice that the scientists were not able to observe and to determine the solar parallax with sufficient accuracy as expected by Halley. Evidence contrary to this claim is not only given by the works of Euler and Duséjour, but by Simon Newcomb (1835-1909) at the end of the 19th century.

The results achieved by Encke and Newcomb

After the beginning of the 19th century the theory of parameter estimation was definitely established by Gauss who provided the mathematical foundations of the method of least squares adjustment. The expert in celestial mechanics, Johann Franz Encke (1791-1865), tried to reprocess all observations acquired from the 1761 and 1769 transit of Venus by using least squares adjustment. There was an important reason for this enterprise. During the first half of the 19th century astrometry, i.e., the measurement of star positions, had been pushed forward immensely, particularly by the observatories of Dorpat, Königsberg and Pulkovo. The instrument makers Reichenbach and Repsold developed and built meridian and transit telescopes of exceptional quality allowing to measure for the first time stellar parallaxes, to prove polar motion, or to make precise stellar catalogues. In this context the accurate determination of the fundamental astronomical constants became an urgent problem which had to be solved with high priority. Apart from the constants of precession, nutation, or aberration, to mention but a few, the re-estimation of the solar parallax was a necessary task. Without knowing high-precision values of these astronomical constants the current problems of that time, particularly the processing of astrometric measurements, would have remained unsolvable. The value for the solar parallax was no longer accurate enough to meet future requirements posed by theory and observation.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Parallaxe du Soleil.

Apogée .... 8",6959.
Moyenne diſtance.................... 8,8418.
Périgée.......................... 8,9931.

Ces parallaxes ſont des parallaxes horizontales polaires, plus petites que celles qui répondent à l'Équateur, dans le rapport de 229 à 230.

Mais

Diſtance du Soleil à la Terre $= \frac{206265}{\text{parallaxe}}$ demi-petit axe de la Terre.

Donc

Diſtances du Soleil à la Terre, évaluées en demi-petit axe de la Terre.

Apogée.......................... 23720.
Moyenne diſtance.................... 23328.
Périgée .... 22936.

Quant au demi-diamètre de Vénus, nous avons vu qu'il étoit égal à 28",345 ; ce demi-diamètre eſt celui qui a été obſervé de la Terre, lorſque ſa diſtance à Vénus égaloit 0,28896; ce même demi-diamètre vu de la diſtance de la Terre au Soleil, auroit donc été obſervé ſous l'angle de $\frac{0,28896}{1,01515}$ 28",345 $=$ 8",068; mais le demi-diamètre polaire de la Terre, vu du Soleil, auroit été obſervé ſous l'angle de 8",710; donc

Demi-diamètre de Vénus $=$ 0,926 demi-diamètre polaire de la Terre.

Détermination de l'heure de la conjonction, du lieu de la conjonction dans l'Écliptique, & de l'erreur des Tables en longitude & en latitude, pour le paſſage du 6 Juin 1761.

(30.) pour déterminer l'heure de la conjonction, le lieu de la conjonction dans l'Écliptique, & l'erreur des Tables, pour le paſſage du 6 Juin 1761, je ferai uſage de l'obſervation de Stockolm; la longitude de cette ville par rapport à Paris, eſt de 1^h 2' 50" orientale.

Figure 30: Page 330 of the treatise published by Duséjour in 1784, showing the final result for the mean solar parallax. (Image: A. Verdun)

Figure 30: Page 330 du traité publié par Duséjour en 1784, montrant le résultat final pour la parallaxe solaire moyenne. (Image: A. Verdun)

Encke presented his results in three treatises which were published in 1822[32], 1824[33], and 1835[34]. He endeavoured to gather all observations available and to prepare them for processing. This task involves the reconstruction of the positions of the observation stations and the precise determination of their geographical coordinates. From both transits he estimated the following values for the solar parallax using the modern methods mentioned above:

Year of publication	Mean solar parallax	Error
1822	8.490525"	± 0.060712"
1824	8.5776"	± 0.0370"
1835	8.57116"	± 0.0370"

Table 5: Encke's results of the mean solar parallax

Table 5: Résultats de Encke pour la parallaxe solaire moyenne

The result of 1835 was valid indubitably for over 20 years. However, in 1854 Encke's colleague and expert in celestial mechanics, Peter Andreas Hansen (1795-1874), pointed out by the parallactic equation of the Moon that the solar parallax must be much larger than the value given by Encke. Using his lunar theory Hansen estimated the value 8.916" for the solar parallax in 1863/64[35]. Pending the imminent transits of Venus of 1874 and 1882 it was expected to definitively solve the problem concerning the true value of the solar parallax, in particular because of the possibility to make use of a newly invented observation technique: photography. This technique allowed for the first time to record the entire progress of a transit photographically and to measure the angular distances between the centres of the Sun's and Venus' disks, thus crucially increasing the number of observations. This is an important aspect due to the fact that the error of an estimated parameter decreases with the square-root of the number of observations. However, the 19th century transits did not yield the expected results: the required increase of accuracy needed to speak of a significantly satisfactory result was simply too high to achieve even with the new observation methods. Nevertheless, Newcomb took pains to process again all observations of the 1761 and 1769 transits. His calculations and results were published in 1891 as part 5 of the second volume of the famous series *Astronomical Papers prepared for the Use of the American Ephemeris and Nautical Almanac*[36]. In the introduction Newcomb discusses possible problems in Encke's treatises, leaving it, however, unquestioned why Encke obtained a value of the solar parallax which was too small: «*The question may be asked, why the final result for the solar parallax obtained in the present paper differs so widely from that deduced by* ENCKE *from the same observations. The completeness and thoroughness of* ENCKE*'s work, with which the writer has been more and more impressed as he proceeded with his own, makes this question all the more pertinent. At the same time he is not prepared to give a definitive answer, for the reason that he has throughout avoided any such comparison of his own work with that of his predecessor as might, by any possibility, bias his judgment in discussing the observations. He entertains the hope that some other astronomer will consider the sub-*

32 Cf. Encke (1822).
33 Cf. Encke (1824).
34 Cf. Encke (1835).
35 Cf. Hansen (1863)
36 Cf. Newcomb (1891).

ject of sufficient interest to make a thorough comparison of the two sets of results.»[37] He mentions possible causes: Inaccurate longitudes of the observation stations, biased weighting of the observations, biased selection of observations, manipulation of observations consequently causing systematic errors (particularly of observations which are supposed to be affected by the black drop phenomenon). Even calculation errors may have deteriorated Encke's result, considering that in his time an adjustment of such magnitude was a rather troublesome and difficult business.

Newcomb's result confirmed the values achieved by Euler and Duséjour: he obtained for the mean solar parallax $\pi_\odot$ = 8.79" with a mean error of ±0.051" and a probable error of ±0.034"[38]. Keeping in mind Halley's claim that an accuracy of 0.02" was feasible (which would have been excellent for those times!) that goal proved more or less to have been achieved by Euler and Duséjour. Newcomb's result of the mean solar parallax coincided with the modern value $\pi_\odot$ = 8.794148" very well. The reason why the 18th century transits of Venus sometimes are judged as a failure may also be found in the steadily increasing accuracy of the solar parallax required for theory, a requirement which in every century was always greater than what the observation and processing methods were able to meet. From the historical point of view the observation campaigns of the 18th century transits of Venus and the development of processing and parameter estimation methods initialized by these events have to be judged as great success.

Conclusions

The observation campaigns performed on the occasion of the transits of Venus in 1761 and 1769 confronted astronomers with a completely new situation. For the first time they were faced by the problem of processing a huge amount of observations from which a very small quantity – the solar parallax – had to be determined. The traditional methods of averaging were totally insufficient to master this task. New parameter estimation methods had to be developed. The procedures used by Euler and Duséjour pointed in the right direction: Their methods of parameter estimation were already very similar to modern adjustment methods. The results obtained by Euler and Duséjour as well as the reprocessing performed by Newcomb, who confirmed their results, prove that the 18th century transits were successful with respect to both the quality of the observations and the development of processing methods initialized by Euler and Duséjour. In fact, the efforts performed in the late 18th century to process the data acquired from the transits of Venus may be seen as the first steps towards the development of modern adjustment and parameter estimation methods.

ANDREAS VERDUN

[37] ibidem, p. 268.
[38] ibidem, p. 402.

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Abbreviations:
H & M: Histoire de l'Académie Royale des Sciences avec les Mémoires de Mathématique et de Physique, Tirés des Registres de cette Académie (Paris)
Phil. Tr.: Philosophical Transactions, giving some Account of the Present Undertakings, Studies and Labours, of the Ingenious, in many Considerable Parts of the World (London)

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